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Wandering Subspaces and Quasi-wandering Subspaces in the Hardy–Sobolev Spaces

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Abstract In this paper, we prove that for $-\frac{1}{2} \leq \beta \leq 0$, suppose M is an invariant subspaces of the Hardy–Sobolev spaces $H^2_{\beta}(\mathbb{D})$ for T^{β}_z , then $M \ominus zM$ is a generating wandering subspace of M, that is, $M = [M \ominus zM]_{T^{\beta}_z}$. Moreover, any non-trivial invariant subspace M of $H^2_{\beta}(\mathbb{D})$ is also generated by the quasi-wandering subspace $P_M T^{\beta}_z M^{\perp}$, that is, $M = [P_M T^{\beta}_z M^{\perp}]_{T^{\beta}_z}$.

Keywords Hardy–Sobolev space, invariant subspace, wandering subspace, quasi-wandering subspace, Beurling type theorem

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1 Introduction

Let \mathbb{D} be the unit disk in complex domain \mathbb{C} , and \mathbb{T} be its boundary. We denote the Lebesgue measure on \mathbb{T} by $d\theta$. For $\beta \in \mathbb{R}$, the Hardy–Sobolev space $H^2_{\beta}(\mathbb{D})$ consists of analytic functions f in \mathbb{D} so that $\mathcal{R}^{\beta}f \in H^2(\mathbb{D})$, where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is the Taylor expansion of f and $\mathcal{R}^{\beta}f = \sum_{k=0}^{\infty} (1+k)^{\beta} a_k z^k$.

We define the norm on $H^2_{\beta}(\mathbb{D})$ as

$$\|f\|_{\beta} = \|\mathcal{R}^{\beta}f\|_{H^2(\mathbb{D})}.$$

In fact, $H^2_{\beta}(\mathbb{D})$ is a Hilbert space with the inner product defined as

$$\langle f,g \rangle_{\beta} = \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{R}^{\beta}(f) \overline{\mathcal{R}^{\beta}(g)} d\theta, \quad \forall f,g \in H^{2}_{\beta}(\mathbb{D}).$$

Definition 1.1 For $\varphi \in H^2_{\beta}(\mathbb{D})$, denote by T^{β}_{φ} the multiplier with symbol φ on $H^2_{\beta}(\mathbb{D})$, that is $T^{\beta}_{\varphi}f = \varphi f$ for any $f \in H^2_{\beta}(\mathbb{D})$. Set

 $\mathcal{M}_{\beta} = \{ \varphi \in H^2_{\beta}(\mathbb{D}) \mid T^{\beta}_{\varphi} \text{ is bounded on } H^2_{\beta}(\mathbb{D}) \}.$

It is not difficult to see that \mathcal{M}_{β} is an algebra and $z \in \mathcal{M}_{\beta}$.

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The Hardy–Sobolev space is a general analytic function space which contains many classical function spaces. For example, it is easy to check that $H_{\frac{1}{2}}^2 = \mathfrak{D}$, the Dirichlet space; $H_0^2 = H^2$, the Hardy space; $H_{-\frac{1}{2}}^2 = L_a^2$, the Bergman space. This means that the Hardy–Sobolev space has very complex constructions. In recent years, a series of papers and books discussed these spaces and the integral operators on these spaces (see [4, 5, 7, 14]).

For a given operator T on a separable Hilbert space H, a closed subspace M of H is called an invariant subspace for T if $TM \subset M$. For an invariant subspace M of H for T, the space $M \ominus TM$ is called the wandering subspace and $P_M TM^{\perp}$ the quasi-wandering subspace respectively for M, where and in what follows P_M denotes the projection onto a closed subspace M and $M^{\perp} = H \ominus M$, $TM^{\perp} = \{Tx : x \in M^{\perp}\}$.

For a subset E of H, we shall denote by $[E]_T$ the smallest invariant subspace of H for T containing E. In other words, $[E]_T$ is the norm-closed linear span of functions of the form $T^k \psi$ for $\psi \in E$ and $k = 0, 1, \ldots$

We say that the Beurling type theorem holds for T if $[M \oplus TM]_T = M$ for all invariant subspaces M of H. On the Hardy space $H^2(\mathbb{D})$, the well known Beurling theorem in [3] says that for all invariant subspaces M of the unilateral shift T_z , their wandering subspaces have dimension 1, and the Beurling type theorem holds for T_z . On the other hand, on the Bergman space $L^2_a(\mathbb{D})$, the situation of the Bergman shift \mathfrak{B} is a little bit different. There are studies of the dimension of wandering subspaces of invariant subspaces of \mathfrak{B} , and it is known that the dimension ranges from 1 to ∞ , see [2, 8, 9]. In 1996, Aleman et al. [1] gave a big progress in the study of invariant subspaces of \mathfrak{B} . They proved the Beurling type theorem for the Bergman shift \mathfrak{B} . This result reveals the inside of the structure of invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on $L^2_a(\mathbb{D})$. Later, different proofs of the Beurling type theorem are given in [13, 15, 16]. In [15], Shimorin proved the following theorem.

Theorem 1.2 (Shimorin's theorem) Let T be a bounded linear operator on a Hilbert space H. If T satisfies the following conditions

- (a) $||Tx + y||^2 \le 2(||x||^2 + ||Ty||^2), \forall x, y \in H.$
- (b) $\bigcap_{n=0}^{\infty} T^n H = \{0\}.$

Then $H \ominus TH$ is a generating wandering subspace for T, that is,

$$H = [H \ominus TH]_T.$$

If T satisfies conditions (a) and (b), then $T|_M : M \to M$ also satisfies conditions (a) and (b). Hence by Shimorin's theorem, the Beurling type theorem holds for T. As an application of this theorem, Shimorin gave a simpler proof of the Aleman, Richter, and Sundberg's theorem. In [16], Sun and Zheng gave another proof of this theorem. Their idea was to lift up the Bergman shift as the compression of a commuting pair of isometries on the subspace of the Hardy space over the bidisk. Sun and Zheng's idea has two aspects. One is to show some identities in the Bergman space. Another one is a technique how to prove the Beurling type theorem.

Recently, Izuchi et al. [10] proved the following theorem.

Theorem 1.3 Let T be a bounded linear operator on a Hilbert space H. If T satisfies the

following conditions

(1) $||Tx||^2 + ||T^{*2}Tx||^2 \le 2||T^*Tx||^2, \forall x \in H.$

(2) T is bounded below, that is, there is c > 0 satisfying $||Tx|| \ge c||x||$ for every $x \in H$.

(3) $||T^{*k}x|| \to 0$ as $k \to \infty$ for every $x \in H$.

Then $H \ominus TH$ is a generating wandering subspace for T, that is,

 $H = [H \ominus TH]_T.$

In [10], it is proved that conditions (1), (2), and (3) in Theorem 2.1 are equivalent to Shimorin's conditions (a) and (b). As application of Theorem 2.1, Izuchi et al. gave an elementary proof of the Aleman, Richter and Sundberg's theorem using some basic function theory in $L^2_a(\mathbb{D})$ and elementary techniques in functional analysis.

In this paper, we study wandering subspaces and quasi-wandering subspaces of the Hardy–Sobolev spaces $H^2_{\beta}(\mathbb{D})$.

2 The Wandering Subspaces of Hardy–Sobolev Spaces

In this section, the following theorem is our main result.

Theorem 2.1 If $-\frac{1}{2} \leq \beta \leq 0$, and M is an invariant subspaces of the Hardy–Sobolev spaces $H^2_{\beta}(\mathbb{D})$ for T^{β}_z , then $M \ominus zM$ is a generating wandering subspace of M, that is,

$$M = [M \ominus zM]_{T_z^\beta}$$

In order to prove Theorem 2.1, we need some identities in the $H^2_{\beta}(\mathbb{D})$. Suppose $f \in H^2_{\beta}(\mathbb{D}), f = \sum_{k=0}^{\infty} a_k z^k$. It is easy to show that

Lemma 2.2 (1) $T_z^{\beta} f = \sum_{k=0}^{\infty} a_k z^{k+1}$. (2) $T_z^{\beta^*} T_z^{\beta} f = \sum_{k=0}^{\infty} \frac{(2+k)^{2\beta}}{(1+k)^{2\beta}} a_k z^k$. (3) $T_z^{\beta^{*2}} T_z^{\beta} f = \sum_{k=1}^{\infty} \frac{(2+k)^{2\beta}}{k^{2\beta}} a_k z^{k-1}$.

By Lemma 2.2 and Theorem 1.3, we prove the main theorem.

Proof of Theorem 2.1 In fact, $||z^k||_{\beta} = (1+k)^{\beta}$, and

$$T_z^{\beta^*} 1 = 0, \quad T_z^{\beta^*}(z^k) = \frac{(1+k)^{2\beta}}{k^{2\beta}} z^{k-1} \quad (k \ge 1).$$

Suppose $f \in H^2_{\beta}$, $f = \sum_{k=0}^{\infty} a_k z^k$. Then $||f||^2_{\beta} = \sum_{k=0}^{\infty} |a_k|^2 (1+k)^{2\beta} < \infty$. Theorem 2.1 follows if we show that the three conditions of Theorem 1.3.

Step 1 We need to prove the following inequality.

$$\|T_{z}^{\beta}f\|_{\beta}^{2} + \|T_{z}^{\beta^{*2}}T_{z}^{\beta}f\|_{\beta}^{2} \le 2\|T_{z}^{\beta^{*}}T_{z}^{\beta}f\|_{\beta}^{2}.$$
(2.1)

It is easy to calculate that

$$\|T_{z}^{\beta}f\|_{\beta}^{2} + \|T_{z}^{\beta^{*2}}T_{z}^{\beta}f\|_{\beta}^{2} = \sum_{k=0}^{\infty} |a_{k}|^{2}(2+k)^{2\beta} + \sum_{k=1}^{\infty} |a_{k}|^{2}\frac{(2+k)^{4\beta}}{k^{4\beta}}k^{2\beta}$$
$$= \sum_{k=0}^{\infty} |a_{k}|^{2}(2+k)^{2\beta} + \sum_{k=1}^{\infty} |a_{k}|^{2}\frac{(2+k)^{4\beta}}{k^{2\beta}}, \qquad (2.2)$$

and

$$2\|T_z^{\beta^*}T_z^{\beta}f\|_{\beta}^2 = 2\sum_{k=0}^{\infty} |a_k|^2 \frac{(2+k)^{4\beta}}{(1+k)^{4\beta}} (1+k)^{2\beta}$$
$$= 2\sum_{k=0}^{\infty} |a_k|^2 \frac{(2+k)^{4\beta}}{(1+k)^{2\beta}}.$$
(2.3)

In order to prove the inequality (2.1), by comparing (2.2) and (2.3), we only need to prove that

$$2^{2\beta} \le 2 \times 2^{4\beta} \tag{2.4}$$

and

$$(2+k)^{2\beta} + \frac{(2+k)^{4\beta}}{k^{2\beta}} \le 2\frac{(2+k)^{4\beta}}{(1+k)^{2\beta}}, \quad \forall k \ge 1.$$
(2.5)

First, it is easy to see that for $-\frac{1}{2} \leq \beta \leq 0$, we have that (2.4) holds.

Next, write $t = 2\beta$, the inequality (2.5) is equal to

$$1 + \frac{(2+k)^t}{k^t} \le 2\frac{(2+k)^t}{(1+k)^t} \Longleftrightarrow \frac{1}{(2+k)^t} + \frac{1}{k^t} \le \frac{2}{(1+k)^t}$$
$$\iff \frac{1}{(2+k)^t} - \frac{1}{(1+k)^t} \le \frac{1}{(1+k)^t} - \frac{1}{k^t}.$$

When $-\frac{1}{2} \leq \beta \leq 0$, that is, $-1 \leq t \leq 0$, we have $\frac{1}{(1+k)^t} \geq \frac{1}{k^t}$. In order to prove the inequality (2.5), we only need to prove that

$$\frac{\frac{1}{(2+k)^t} - \frac{1}{(1+k)^t}}{\frac{1}{(1+k)^t} - \frac{1}{k^t}} \le 1.$$
(2.6)

For $-1 \le t \le 0$, since the functions $g(x) = (x + k + 1)^{-t}$ and $h(x) = (x + k)^{-t}$ is derivable on the interval [0, 1], by the Cauchy mean value theorem, there exists a $\xi \in [0, 1]$ such that

$$\frac{g(1) - g(0)}{h(1) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{-t(1+k+\xi)^{-t-1}}{-t(k+\xi)^{-t-1}} = \frac{(k+\xi)^{t+1}}{(1+k+\xi)^{t+1}} \le 1,$$

and

$$\frac{g(1) - g(0)}{h(1) - h(0)} = \frac{g'(\xi)}{h'(\xi)} = \frac{\frac{1}{(2+k)^t} - \frac{1}{(1+k)^t}}{\frac{1}{(1+k)^t} - \frac{1}{k^t}}.$$

This means that (2.6) follows, then the inequality (2.5) follows.

Hence we have that for $-\frac{1}{2} \leq \beta \leq 0$,

$$\|T_{z}^{\beta}f\|_{\beta}^{2} + \|T_{z}^{\beta^{*2}}T_{z}^{\beta}f\|_{\beta}^{2} \le 2\|T_{z}^{\beta^{*}}T_{z}^{\beta}f\|_{\beta}^{2}.$$

Step 2 For $\beta \leq 0$, it is easy to see that

$$||T_z^\beta f||_\beta \ge 2^\beta ||f||_\beta.$$

That is, T_z is bounded below.

Step 3 We will prove that for $\beta \leq 0$,

$$\lim_{n \to \infty} \|T_z^{\beta^{*n}} f\|_{\beta} = 0, \quad \forall f \in H_{\beta}^2(\mathbb{D}).$$

In fact, for n > k, we have $T_z^{\beta^{*n}}(z^k) = 0$; for $n \le k$, we have $T_z^{\beta^{*n}}(z^k) = \frac{(1+k)^{2\beta}}{(1+k-n)^{2\beta}} z^{k-n}$. By calculating,

$$\begin{split} \|T_z^{\beta^{*n}}f\|_{\beta}^2 &= \left\|\sum_{k=n}^{\infty} |a_k|^2 \frac{(1+k)^{2\beta}}{(1+k-n)^{2\beta}} z^{k-n}\right\|_{\beta}^2 \\ &= \sum_{k=n}^{\infty} |a_k|^2 \frac{(1+k)^{4\beta}}{(1+k-n)^{4\beta}} \|z^{k-n}\|_{\beta}^2 \\ &= \sum_{k=n}^{\infty} |a_k|^2 \frac{(1+k)^{4\beta}}{(1+k-n)^{2\beta}}. \end{split}$$

Since $f \in H_{\beta}^2$, $f = \sum_{k=0}^{\infty} a_k z^k$, $||f||_{\beta}^2 = \sum_{k=0}^{\infty} |a_k|^2 (1+k)^{2\beta} < \infty$. Then, when $n \to \infty$, we get $\sum_{k=n}^{\infty} |a_k|^2 (1+k)^{2\beta} \to 0$. For $\beta \le 0$, note that $\frac{(1+k)^{2\beta}}{(1+k-n)^{2\beta}} \le 1 \ (k \ge n)$. We have

$$||T_z^{\beta^{*n}} f||_{\beta}^2 = \sum_{k=n}^{\infty} |a_k|^2 \frac{(1+k)^{4\beta}}{(1+k-n)^{2\beta}}$$
$$\leq \sum_{k=n}^{\infty} |a_k|^2 (1+k)^{2\beta} \to 0.$$

Hence we get that for $\beta \leq 0$, $\lim_{n \to \infty} ||T_z^{\beta^{*n}} f||_{\beta} = 0$.

In conclusion, T_z^β satisfies the three conditions of Theorem 1.3, then $M \ominus zM$ is a generating wandering subspace of M, that is,

$$M = [M \ominus zM]_{T^{\beta}}.$$

3 The Quasi-wandering Subspaces of Hardy–Sobolev Spaces

An operator T on a Hilbert space H has the quasi-wandering property if for each nontrivial invariant subspace M of H for T, $M = [P_M T M^{\perp}]_T$.

Let M be an nontrivial invariant subspace of $H^2(\mathbb{D})$ for T_z . Since in this case T_z is an isometry, i.e., $T_z^*T_z = I$, then one easily sees $P_M T_z M^{\perp} \subset M \ominus T_z M$. On the other hand, it is easy to check (see [11])

$$\dim M \ominus T_z M \le \dim P_M T_z M^{\perp}. \tag{3.1}$$

Thus we have

$$P_M T_z M^\perp = M \ominus z M.$$

So in the one variable Hardy space case, a quasi-wandering subspace coincides with a wandering subspace for M. It is not the case in the Bergman space although (3.1) still holds. The quasi-wandering subspace in the Bergman space is studied recently by Izuchi et al. in [11], they proved the following theorem.

Theorem 3.1 Let M be an invariant subspace of $L^2_a(\mathbb{D})$ for T_z . Then the following conditions are equivalent:

- (1) $P_M T_z M^{\perp} is not dense in M$.
- (2) There exists an $f \in M$ with $f \neq 0$ satisfying $T_z^* f \in M$.
- (3) $M \cap \mathfrak{D} \neq \{0\}, \mathfrak{D}$ is Dirichlet space.
- (4) There exists an $f \in M$ with $f \neq 0$ satisfying $f' \in M$.
- (5) There exists an $f \in M$ with $f \neq 0$ satisfying $f T_z T_z^* f \in M$.

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Moreover, in [11], it is shown that the shift operator on the Bergman space $L^2_a(\mathbb{D})$ has the quasi-wandering property. This result is seen as a counterpart of the Aleman, Richter and Sundberg's theorem.

Theorem 3.2 ([11]) Let M be an nontrivial invariant subspace of $L^2_a(\mathbb{D})$ for T_z . Then $M = [P_M T_z M^{\perp}]_{T_z}$.

Recently, Chen [6] considers the quasi-wandering property for *n*-tuple of operators $\overline{T} = (T_{z_1}, \ldots, T_{z_n})$ on a reproducing analytic Hilbert space H_n^K on the unit ball \mathbb{B}_n of \mathbb{C}^n defined by a \mathcal{U} -invariant kernel K.

We say a closed subspace M of Hilbert space H is invariant for an *n*-tuple of operators $\overline{T} = (T_1, \ldots, T_n)$, if $T_i M \subset M$ for all $i = 1, \ldots, n$. Set

$$P_M \overline{T} M^{\perp} = P_M T_1 M^{\perp} + \dots + P_M T_n M^{\perp}.$$

We call an operator tuple $\overline{T} = (T_1, \ldots, T_n)$ has the quasi-wandering property if for each nontrivial invariant subspace M of H for \overline{T} , $M = [P_M \overline{T} M^{\perp}]_{\overline{T}}$.

Fix a constant v with v > 0, set

$$K_v(z,w) = \frac{1}{(1 - \langle z, w \rangle)^v}.$$

Chen proved the following theorem.

Theorem 3.3 ([6]) If $K = K_v$ or K is a \mathcal{U} -invariant complete Nevanlinna-Pick kernel, then the n-shift $\overline{T} = (T_{z_1}, \ldots, T_{z_n})$ on H_n^K has the quasi-wandering property.

Motivated by the work of Chen, in this section we will consider the quasi-wandering property for T_z on the Hardy–Sobolev Spaces $H^2_{\beta}(\mathbb{D})$.

The following lemmas are very useful in our proof.

Lemma 3.4 Let M be an invariant subspace of a Hilbert space H for T. Then $M \ominus [P_M T M^{\perp}]_T$ = $H \ominus [M^{\perp}]_T$. Moreover, $[P_M T M^{\perp}]_T = M$ if and only if $[M^{\perp}]_T = H$. Proof Suppose $f \in H \ominus [M^{\perp}]_T$. Then

$$f \perp T^k M^\perp$$

for all $k \ge 0$. Obviously $f \in M$. Since

$$T^{k}P_{M}TM^{\perp} \subset T^{k}TM^{\perp} - T^{k}P_{M^{\perp}}TM^{\perp}$$
$$\subset T^{k+1}M^{\perp} - T^{k}M^{\perp},$$

we have $f \perp T^k P_M T M^{\perp}$ for all $k \geq 0$. Hence $f \in M$ and $f \perp [P_M T M^{\perp}]_T$, that is, $f \in M \oplus [P_M T M^{\perp}]_T$. For the converse, suppose $f \in M \oplus [P_M T M^{\perp}]_T$. Then

$$f \in M, \quad f \perp T^k P_M T M^{\perp}, \quad k \ge 0. \tag{3.2}$$

Noting that for $k \ge 1$,

$$\begin{split} T^k M^{\perp} &\subset T^{k-1} P_M T M^{\perp} + T^{k-1} P_{M^{\perp}} T M^{\perp} \\ &\subset T^{k-1} P_M T M^{\perp} + T^{k-1} M^{\perp}, \end{split}$$

then by induction, we have

$$T^k M^{\perp} \subset \sum_{i=1}^k T^{k-i} P_M T M^{\perp} + M^{\perp}.$$

It follows from (3.2) that $f \perp T^k M^{\perp}$ for all $k \ge 0$. We get that $f \in H \ominus [M^{\perp}]_T$. The proof is complete.

In the last part of this paper, we consider the quasi-wandering property for T_z on the Hardy–Sobolev Spaces $H^2_{\beta}(\mathbb{D})$.

Since

$$e_n(z) = \frac{z^n}{(1+n)^\beta}, \quad n \ge 0$$

forms a canonical orthonormal basis for Hilbert space $H^2_\beta(\mathbb{D})$, then the reproducing kernel of $H^2_\beta(\mathbb{D})$ is

$$K(z,w) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{(1+n)^{2\beta}}.$$

Since K(0,0) = 1, in some open neighborhood of zero, we have

$$\frac{1}{K(z,w)} = \sum_{k=0}^{\infty} c_k z^k \overline{w}^k.$$

Because

$$\mathbf{l} = \frac{1}{K(z,w)} K(z,w) = \left(\sum_{k=0}^{\infty} c_k z^k \overline{w}^k\right) \left(\sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{(1+n)^{2\beta}}\right),$$

we obtain that

$$\begin{cases} c_0 = 1, \\ \sum_{k \le n} \frac{c_k}{(1+n-k)^{2\beta}} = 0, \quad n \ge 1. \end{cases}$$
(3.3)

Set $S = \sum_{k=0}^{\infty} c_k T_z^{\beta^k} T_z^{\beta^{*k}}$, $S_N = \sum_{k=0}^{N} c_k T_z^{\beta^k} T_z^{\beta^{*k}}$. It is not difficult to check that

$$T_z^{\beta^k} T_z^{\beta^{*k}}(z^n) = \begin{cases} 0, & n < k, \\ \frac{(1+n)^{2\beta}}{(1+n-k)^{2\beta}} z^n, & n \ge k. \end{cases}$$
(3.4)

Lemma 3.5 For every polynomial P of $H^2_{\beta}(\mathbb{D})$, S(P) = P(0). Hence, S can be extended continuously on $H^2_{\beta}(\mathbb{D})$ as the evaluation functional at zero.

Proof When $n \ge 1$, by (3.3) and (3.4), for every monomial z^n ,

$$S(z^{n}) = \sum_{k=0}^{\infty} c_{k} T_{z}^{\beta^{k}} T_{z}^{\beta^{*}k}(z^{n})$$
$$= \sum_{k \leq n} c_{k} T_{z}^{\beta^{k}} T_{z}^{\beta^{*}k}(z^{n})$$
$$= \sum_{k \leq n} c_{k} \frac{(1+n)^{2\beta}}{(1+n-k)^{2\beta}} z^{n}$$

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$$= (1+n)^{2\beta} z^n \sum_{k \le n} \frac{c_k}{(1+n-k)^{2\beta}}$$

= 0

then S(P) = P(0), Hence, S can be extended continuously on $H^2_{\beta}(\mathbb{D})$ as the evaluation functional at zero.

Lemma 3.6 When $N \to \infty$, $S_N \to S(SOT)$ in $H^2_{\beta}(\mathbb{D})$. That is, $\|S_N f - Sf\|_{H^2_{\beta}}(\mathbb{D}) \to 0, \forall f \in H^2_{\beta}(\mathbb{D})$.

Proof Fix each $f \in H^2_{\beta}, \forall \varepsilon > 0$, there exists a polynomial P_n such that

$$\|f - P_n\|_{\beta} < \varepsilon.$$

Set N > n, a direct calculation gives that

$$\begin{aligned} \|(S - S_N)f\|_{\beta} &\leq \|(S - S_N)P_n\|_{\beta} + \|(S - S_N)(f - P_n)\|_{\beta} \\ &= P_n(0) - P_n(0) + \|(S - S_N)(f - P_n)\|_{\beta} \\ &\leq \|S - S_N\|\|f - P_n\|_{\beta} \\ &\leq (\|S\| + \|S_N\|)\|f - P_n\|_{\beta} \\ &\leq M\varepsilon, \end{aligned}$$

where M is a constant. Then we have $S_N \to S(SOT)$.

We are now ready to prove the main result.

Theorem 3.7 Let M be a nontrivial invariant subspace of $H^2_\beta(\mathbb{D})$ for T^β_z . Then $M = [P_M T^\beta_z M^{\perp}]_{T^{\beta}_z}$.

Proof In order to prove that $M = [P_M T_z^{\beta} M^{\perp}]_{T_z^{\beta}}$, by Lemma 3.4, we only need to prove that $H_{\beta}^2 = [M^{\perp}]_{T_z^{\beta}}$. If the claim is not true, that is, $H_{\beta}^2 \neq [M^{\perp}]_{T_z^{\beta}}$, then there exists $f \in H_{\beta}^2$ with $f \neq 0$, and $f \perp [M^{\perp}]_{T_z^{\beta}}$. Since $f \perp T_z^{\beta k} M^{\perp} (k \ge 0)$, we have $T_z^{\beta * k} f \in M$. Since M is an invariant subspace, then

$$T_z^{\beta^k} T_z^{\beta^{*k}} f \in M \ (k \ge 0).$$

We write $f = \sum_{k=0}^{\infty} a_k z^k$ with some $a_i \neq 0$. Let $h = T_z^{\beta^{*i}} f \in M$, $h = \sum_{k=0}^{\infty} b_k z^k$, where $b_0 \neq 0$. Since for all N, $S_N h = \sum_{k=0}^{N} c_k T_z^{\beta^k} T_z^{\beta^{*k}} h \in M$, by Lemma 3.6, we have $||S_N h - Sh|| \to 0$. Noting that M is closed, by Lemma 3.5, we have $Sh = b_0 \in M$, which implies that $H_{\beta}^2 = M$, a contradiction. So the claim holds, the proof is completed.

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References

- Aleman, A., Richter, S., Sundberg, C.: Beurlings theorem for the Bergman space. Acta Math., 117, 275–310 (1996)
- [2] Apostol, C., Das, B., Sarkar, J., et al.: Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. J. Funct. Anal., 63, 369–404 (1985)
- Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math., 81, 239–255 (1949)
- [4] Cao, G. F., He, L.: Hardy–Sobolev spaces and their multipliers. Sci. China Math., 57, 23610–2368 (2014)

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- [5] Cao, G. F., He, L.: Fredholmness of multipliers on Hardy–Sobolev spaces. J. Math. Anal. Appl., 418, 1–10 (2014)
- Chen, Y.: Quasi-wandering subspaces in a class of reproducing analytic Hilbert spaces. Proc. Amer. Math. Soc., 140, 4235–4242 (2012)
- [7] Cho, H. R., Zhu, K. H.: Holomorphic mean Lipschitz spaces and Hardy–Sobolev spaces on the unit ball. Complex Variables and Elliptic Equations, 57, 995–1024 (2012)
- [8] Hedenmalm, H.: An invariant subspace of the Bergman space having the codimension two property. J. Reine Angew. Math., 443, 1–9 (1993)
- Hedenmalm, H., Richter, S., Seip, K.: Interpolating sequences and invariant subspaces of given index in the Bergman spaces. J. Reine Angew. Math., 477, 13–30 (1996)
- [10] Izuchi, K. J., Izuchi, K. H., Izuchi, Y.: Wandering subspaces and the Beurling type theorem I. Arch. Math., 95, 439–446 (2010)
- [11] Izuchi, K. J., Izuchi, K. H., Izuchi, Y.: Quasi-wandering subspaces in the Bergman space. Integr. Equ. Oper. Theory, 67, 151–161 (2010)
- Mccullough, S., Richter, S.: Bergman-type reproducing kernels, contractive divisors, and dilations. J. Funct. Anal., 190, 447–480 (2002)
- [13] Olofsson, A.: Wandering subspace theorems. Integr. Equ. Oper. Theory, 51, 395–409 (2005)
- [14] Ortega, J. M., Fabrega, J.: Multipliers in Hardy–Sobolev spaces. Integr. Equ. Oper. Theory, 55, 535–560 (2006)
- [15] Shimorin, S.: Wold-type decompositions and wandering subspaces for operators close to isometries. J. Reine Angew. Math., 531, 147–189 (2001)
- [16] Sun, S., Zheng, D.: Beurling type theorem on the Bergman space via the Hardy space of the bidisk. Sci. China Ser. A, 52, 2517–2529 (2009)