Acta Mathematica Sinica, English Series
Dec., 2017, Vol. 33, No. 12, pp. 1693–1704
Published online: September 29, 2017
https://doi.org/10.1007/s10114-017-6504-9
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Cluster Structures in 2-Calabi–Yau Triangulated Categories of Dynkin Type with Maximal Rigid Objects

Hui Min CHANG
Department of Mathematical Sciences, Tsinghua University,
Beijing 100084*, P. R. China*
and
Department of Applied Mathematics, School of Education, The Open University of China,
Beijing 100039*, P. R. China*
E-mail: *changhm@crtvu.edu.cn*

Abstract In this paper, we consider two kinds of 2-Calabi–Yau triangulated categories with finitely many indecomposable objects up to isomorphisms, called $\mathcal{A}_{n,t} = D^b(\mathbb{K}A_{(2t+1)(n+1)-3})/\tau^{t(n+1)-1}[1]$, where $n, t \geq 1$, and $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_{2t(n+1)})/\tau^{(n+1)}\varphi^n$, where $n, t \geq 1$, and φ is induced by an automorphism of $D_{2t(n+1)}$ of order 2. Except the categories $\mathcal{A}_{n,1}$, they all contain non-zero maximal rigid objects which are not cluster tilting. $\mathcal{A}_{n,1}$ contain cluster tilting objects. We define the cluster complex of $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$) by using the geometric description of cluster categories of type A (resp. type D). We show that there is an isomorphism from the cluster complex of $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$) to the cluster complex of root system of type B_n. In particular, the maximal rigid objects are isomorphic to clusters. This yields a result proved recently by Buan–Palu–Reiten: Let $\mathcal{R}_{\mathcal{A}_{n,t}}$, resp. $\mathcal{R}_{\mathcal{D}_{n,t}}$, be the full subcategory of $\mathcal{A}_{n,t}$, resp. $\mathcal{D}_{n,t}$, generated by the rigid objects. Then $\mathcal{R}_{\mathcal{A}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$ and $\mathcal{R}_{\mathcal{D}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$ as additive categories, for all $t \geq 1$.

Keywords 2-Calabi–Yau triangulated category, cluster structure, cluster complex

MR(2010) Subject Classification 18D35, 18F99, 18B99

1 Introduction

Cluster algebras were first introduced by Fomin and Zelevinsky [10], and further developed in a series of papers [1, 11, 13]. The connection with representation of algebras was first discovered in [19], and then cluster categories were defined in [4], as a categorical version of cluster algebras. Cluster categories were associated with certain orbit categories and shown to be triangulated in [16]. The cluster categories of finite type A_n were studied in [7]. There is a bijection between the indecomposable objects and diagonals of $(n+3)$-gon. Let Q be a finite quiver with no oriented cycles, and $\mathcal{C}_Q$ be the cluster category associated to Q. Then the indecomposable rigid objects are in bijection with the cluster variables in the cluster algebra $\mathcal{A}_Q$ associated to the same quiver, and under this bijection the clusters correspond to the maximal rigid objects [8].

The cluster categories and the stable categories $\underline{\mathrm{mod}}\Lambda$ of preprojective algebras are both 2-Calabi–Yau triangulated categories, and have cluster tilting objects or subcategories [4, 15, 16],

Received October 31, 2016, accepted February 27, 2017
Supported by the NSF of China (Grant No. 11671221)

which are important related to clusters of some cluster algebras. Inspired by this, cluster tilting objects were investigated in [3], and the definition of cluster structure was given. It was shown that the collection of maximal rigid objects in any finite Hom-finite 2-Calabi–Yau triangulated category has a cluster structure if the quivers of their endomorphism rings do not have loops or 2-cycles. The cluster-tilting objects are closely related to a class of objects called maximal rigid objects. Indeed, cluster-tilting objects are maximal rigid objects, but the converse is not true in general [2, 5, 17]. For a connected 2-Calabi–Yau category, either all maximal rigid objects are cluster tilting, or none of them are [23]. And if for a maximal rigid object M there are no loops or 2-cycles in the quiver of $\mathrm{End}(M)$, then M is cluster tilting [2, 21].

The concept of cluster structure was generalized in [5]. If the endomorphism rings of maximal rigid objects in a 2-Calabi–Yau triangulated category may have loops but no 2-cycles, then the collection of maximal rigid objects has a cluster structure. For example, the cluster category of a tube is a 2-Calabi–Yau triangulated category with non-zero maximal rigid objects which are not cluster tilting, and the endomorphism ring of any maximal rigid object has no 2-cycles but has a loop. The collection of maximal rigid objects in cluster tube has a cluster structure in the sense of [5], but does not satisfy the definition of the cluster structure in [3].

In this paper, we investigate two kinds of 2-Calabi–Yau triangulated categories with non-zero maximal rigid objects which are not cluster tilting, called $\mathcal{A}_{n,t} = D^b(\mathbb{K}A_{(2t+1)(n+1)-3})/\tau^{t(n+1)-1}$ [1], where $n \geq 1$ and $t > 1$, and $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_{2t(n+1)})/\tau^{(n+1)}\varphi^n$, where $n, t \geq 1$, and φ is induced by an automorphism of $D_{2t(n+1)}$ of order 2 [6]. Note that $\mathcal{A}_{n,1}$ are finite 2-Calabi–Yau triangulated categories with cluster tilting objects [6]. We show that the set of maximal rigid objects in $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$) forms a cluster structure in the sense of [5]. Using the geometric description of cluster categories of type A (resp. type D), we show that the indecomposable rigid objects in $\mathcal{A}_{n,t}$ (contain $\mathcal{A}_{n,1}$) (resp. $\mathcal{D}_{n,t}$) are in bijection with the cluster variables of a cluster algebra of B_n, and under this bijection, the maximal rigid objects correspond to clusters. This yields a result proved recently in [6]: Let $\mathcal{R}_{\mathcal{A}_{n,t}}$, resp. $\mathcal{R}_{\mathcal{D}_{n,t}}$, be the full subcategory of $\mathcal{A}_{n,t}$, resp. $\mathcal{D}_{n,t}$, generated by the rigid objects. Then $\mathcal{R}_{\mathcal{A}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$ and $\mathcal{R}_{\mathcal{D}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$ as additive categories, for all $t \geq 1$.

This paper is organized as follows: In Section 2, we recall the definition of cluster structure in the sense of [5] and some related results. In Section 3, we show that the set of maximal rigid objects in $\mathcal{A}_{n,t}$ forms a cluster structure in the first subsection, and do the same thing for $\mathcal{D}_{n,t}$ in the second subsection. In Section 4, we define the cluster complex of $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$), denoted by $\Delta(\mathcal{A}_{n,t})$ (resp. $\Delta(\mathcal{D}_{n,t})$), and show that there is an isomorphism from $\Delta(\mathcal{A}_{n,t})$ (resp. $\Delta(\mathcal{D}_{n,t})$) to the cluster complex of root system of type B_n. In particular, the maximal rigid objects are isomorphic to clusters.

Notation Unless stated otherwise, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. Our categories will be assumed to be $\mathbb{K}$-linear, Hom-finite, Krull-Remark-Schmidt additive categories. We say $\mathcal{C}$ is a finite triangulated category if it contains only finitely many indecomposable objects up to isomorphisms. In a triangulated category, we use $\mathrm{Ext}^1(X, Y)$ to denote $\mathrm{Hom}(X, Y[1])$, where $[1]$ is the shift functor of the triangulated category.

2 Cluster Structures

Definition 2.1 *A triangulated category $\mathcal{C}$ is called* 2*-Calabi–Yau* (*shortly* 2-CY) *provided there is a functorially isomorphism*

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{C}}(Y,X[2])$, *for all $X, Y \in \mathcal{C}$, where $D = \mathrm{Hom}_{\mathbb{K}}(-,\mathbb{K})$.*

Definition 2.2 *Let $\mathcal{C}$ be a triangulated category. Suppose $\mathcal{T}$ is a subcategory of $\mathcal{C}$, and T is an object in $\mathcal{C}$.*

(1) *$\mathcal{T}$ is called rigid if $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T},\mathcal{T}) = 0$. $\mathcal{T}$ is called maximal rigid if $\mathcal{T}$ is maximal with respect to this property, i.e., if $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T} \oplus \mathrm{add}\ M, \mathcal{T} \oplus \mathrm{add}\ M) = 0$, then $M \in \mathrm{add}\ T$.*

(2) *T is called a rigid object if* add T *is rigid. T is maximal rigid if* add T *is maximal rigid.*

(3) *A functorially finite subcategory $\mathcal{T}$ is called cluster tilting if $\mathcal{T} = \{X \in \mathcal{C} | \mathrm{Ext}^1_{\mathcal{C}}(X,\mathcal{T}) = 0\} = \{X \in \mathcal{C} | \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{T},X) = 0\}$. An object T is a cluster tilting object if* add T *is a cluster tilting subcategory.*

Lemma 2.3 ([6, Proposition 2.2]) *The standard, finite* 2-CY*, triangulated categories with non-zero maximal rigid objects which are not cluster tilting are exactly the orbit categories*:

- (*Type A*) $\mathcal{A}_{n,t} = D^b(\mathbb{K}A_{(2t+1)(n+1)-3})/\tau^{t(n+1)-1}[1]$, *where $n \geq 1$ and $t > 1$*;
- (*Type D*) $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_{2t(n+1)})/\tau^{(n+1)}\varphi^n$, *where $n, t \geq 1$, and where φ is induced by an automorphism of $D_{2t(n+1)}$ of order* 2;
- (*Type E*) $D^b(\mathbb{K}E_7)/\tau^2$ *and* $D^b(\mathbb{K}E_7)/\tau^5$.

In this paper, we mainly consider the categories $\mathcal{A}_{n,t}$ and $\mathcal{D}_{n,t}$. Note that $\mathcal{A}_{n,1}$ are 2-CY triangulated categories with cluster tilting objects when $t = 1$ [6], and every cluster tilting object is maximal rigid by definition. The categories we considered below include the case in $\mathcal{A}_{n,t}$ with $t = 1$.

Let $\mathcal{C}$ be a finite 2-CY triangulated category of Dynkin type with maximal rigid objects, and denote the set of maximal rigid objects in $\mathcal{C}$ by $\mathcal{T}$. We recall the definition of cluster structure based on [5], We say $\mathcal{T}$ has a *weak cluster structure* if the following conditions are satisfied:

(a) For each $T \in \mathcal{T}$, $T = T_1 \oplus \cdots \oplus T_n$ with all T_i indecomposable. For an $i \in \{1, \ldots, n\}$, write $\bar{T} = \bigoplus_{j \neq i} T_j$, there exists a unique indecomposable object $T_i^* \ncong T_i$ such that $T^* = \bar{T} \oplus T_i^*$ is in $\mathcal{T}$.

(b) T_i and T_i^* are related by two triangles $T_i^* \xrightarrow{f_i} U_i \xrightarrow{g_i} T_i \to$ and $T_i \xrightarrow{s_i} U_i' \xrightarrow{t_i} T_i^* \to$ where f_i and s_i are minimal left add $\bar{T}$-approximations, and g_i and t_i are minimal right add $\bar{T}$-approximations. The two triangles above are called exchange triangles.

Assume that $\mathcal{T}$ has a weak cluster structure. For each T in $\mathcal{T}$, we define a matrix $B_T = (b_{ij})$ as follows:

$$b_{ij} = \alpha'_{ij} - \alpha_{ij},$$

where α'_{ij} denotes the multiplicity of T_j as a direct summand of U_i' and α_{ij} denotes the multiplicity of T_j as a direct summand of U_i.

For a matrix $B = (b_{ij})$, recall Fomin–Zelevinsky *matrix mutation.* For any $k \in \{1, 2, \ldots, n\}$, the mutation $\mu_k(B) = (b'_{ij})$ of B at direction k is given by

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

Definition 2.4 ([5]) *The notations are as above. We say $\mathcal{T}$ has a cluster structure, if $\mathcal{T}$ has a weak cluster structure, and in addition the following conditions are met*:

(c) U_i *and* U_i' *have no common direct summands.*

(d) B_T *and* B_{T^*} *are related by Fomin–Zelevinsky matrix mutation at* i.

In this case, we call the elements of $\mathcal{T}$ *clusters*. The condition (c) above can be interpreted as the endomorphism rings of the clusters have Gabriel quivers which do not have 2-cycles.

Proposition 2.5 ([5]) *Let $\mathcal{C}$ be a Hom-finite* 2-CY *triangulated category, and let $\mathcal{T}$ be the collection of maximal rigid objects in $\mathcal{C}$. Assume that $\mathcal{T}$ is non-empty, and the endomorphism rings of the clusters have no* 2*-cycles. Then $\mathcal{T}$ has a cluster structure.*

3 Cluster Structures in $\mathcal{A}_{n,t}$ and $\mathcal{D}_{n,t}$

In this section, we show the set of maximal rigid objects in $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$) has a cluster structure.

3.1 Cluster Structures in $\mathcal{A}_{n,t}$

Let $\mathcal{C}_{A_{N-3}}$ be the cluster category of type A_{N-3}, where $N = (2t+1)(n+1)$. By the universal property of orbit categories [16], also by the proof of Lemma 2.4 in [6], we know that there exists a covering functor $\pi\colon \mathcal{C}_{A_{N-3}} \to \mathcal{A}_{n,t}$. Write $F = \tau^{t(n+1)}$, then $F : \mathcal{C}_{A_{N-3}} \to \mathcal{C}_{A_{N-3}}$ is an autoequivalence. Since $\tau^{N-2} = [-2]$ in $D^b(A_{N-3})$ by [16] and $\tau = [1]$ in $\mathcal{C}_{A_{N-3}}$, τ is of order N and τ^{n+1} is of order $2t+1$ in $\mathcal{C}_{A_{N-3}}$. Moreover, $\gcd(t, 2t+1) = 1$ implies that $F = \tau^{t(n+1)}$ is also of order $2t+1$, so the groups generated by F and by τ^{n+1} are the same, i.e., $\langle F \rangle = \langle \tau^{n+1} \rangle$. Therefore, $\mathcal{A}_{n,t}$ can be seen as the orbit category $\mathcal{C}_{A_{N-3}}/\tau^{n+1}$.

Let P be an N-gon with a distinguished oriented edge, where $N = (2t+1)(n+1)$. A *diagonal* is a set of two non-neighbouring vertices $\{\alpha, \beta\}$. Two diagonals $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ *cross* if their end points are all distinct and come in the order α_1, β_1, α_2, β_2 when moving around the polygon in one direction or the other.

There is a bijection between indecomposable objects of $\mathcal{C}_{A_{N-3}}$ and diagonals of N-gon P. The Auslander–Reiten translation τ acts on diagonal is rotation by one vertex [7]. We have

$$\dim \operatorname{Ext}^1_{\mathcal{C}_{A_{N-3}}}(a, b) = \begin{cases} 1, & \text{if } a \text{ and } b \text{ cross}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}_{A_{N-3}}$ has only finitely many indecomposable objects, any subcategory of $\mathcal{C}_{A_{N-3}}$ closed under direct sums and direct summands is completely determined by the set of indecomposable objects it contains. Then the bijection between indecomposable objects of $\mathcal{C}_{A_{N-3}}$ and diagonals of P extends to a bijection between subcategories of $\mathcal{C}_{A_{N-3}}$ closed under direct sums and direct summands and sets of diagonals of P.

Definition 3.1 (1) *Let $\mathscr{X}$ be a subcategory of a triangulated category $\mathcal{C}$, and $F\colon \mathcal{C} \to \mathcal{C}$ be an autoequivalence. $\mathscr{X}$ is called an F-periodic if $F\mathscr{X} = \mathscr{X}$.*

(2) *Let $\mathfrak{U}$ be a set of diagonals in the N-gon P. Fix a positive integer $k|N$, and $N = k\ell$ for some integer ℓ. $\mathfrak{U}$ is called a k-periodic collection of diagonals of P if for each diagonal $(i, j) \in \mathfrak{U}$, all diagonals $(i + kr, j + kr)$ (modulo N) for $1 \le r \le \ell$ are in $\mathfrak{U}$.*

For any subcategory $\mathscr{X}$ in $\mathcal{A}_{n,t}$, the preimage under the covering functor π corresponds to an F-periodic subcategory $\widetilde{\mathscr{X}} = \pi^{-1}(\mathscr{X})$ in $\mathcal{C}_{A_{N-3}}$. Moreover, the subcategory $\widetilde{\mathscr{X}}$ corresponds to a set of diagonals of N-gon P by the discussion above, we still denote the corresponding set of diagonals by $\widetilde{\mathscr{X}}$. Since $F = \tau^{n+1}$, the corresponding set of diagonals is $(n + 1)$-periodic. In the rest of the paper, we always use a coordinate (i, j) to represent an indecomposable object of $\mathcal{C}_{A_{N-3}}$ and $[(i, j)]$ to represent the image under the functor π.

Lemma 3.2 *There is a bijection between the following sets:*

(1) *Subcategories $\mathscr{X}$ of $\mathcal{A}_{n,t}$;*

(2) *Collections of diagonals $\widetilde{\mathscr{X}}$ of the N-gon P which are $(n + 1)$-periodic.*

Lemma 3.3 ([6, Lemma 2.4]) (1) *The isomorphism classes of indecomposable rigid objects in $\mathcal{A}_{n,t}$ are parametrised by the arcs $[(i, i + 2)], \dots, [(i, i + n + 1)]$ for $i = 1, \dots, n + 1$.*

(2) *The maximal non-crossing $(n + 1)$-periodic collections of $(2t + 1)(n + 1)$-gon correspond to isoclasses of basic maximal rigid objects.*

The object $[(1, 3)] \oplus \cdots \oplus [(1, n + 2)]$ is a maximal rigid object, its endomorphism algebra is the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \cdots\cdots n-1 \longrightarrow n \circlearrowleft \alpha$$

modulo the ideal generated by α^2, by Corollary 2.7 in [6]. For this maximal rigid object, the endomorphism ring of Gabriel quiver does not have 2-cycles. More generally, we show that it holds for any maximal rigid object in $\mathcal{A}_{n,t}$ in the following.

Theorem 3.4 *The set of maximal rigid objects in $\mathcal{A}_{n,t}$ has a cluster structure.*

Proof By Proposition 2.5 and the fact $\mathcal{A}_{n,t}$ is a Hom-finite 2-Calabi–Yau triangulated category, we only need to show that for any maximal rigid object T in $\mathcal{A}_{n,t}$, there are no 2-cycles in the quiver of $\mathrm{End}_{\mathcal{A}_{n,t}}(T)$. Equivalently, let X and Y be two non-isomorphic indecomposable summands of T, we have to show it is impossible that $\mathrm{Hom}_{\mathcal{A}_{n,t}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{A}_{n,t}}(Y, X)$ are both non-zero.

Without loss of generality, suppose T is a maximal rigid object with indecomposable summand $T_1 = [(1, n+2)]$ of maximal length. Note that for any indecomposable rigid object $[(i, j)]$, we can always choose a representative (i, j) such that $1 \le i \le n+1$, and $3 \le j \le 2(n+1)$. Since X and Y are indecomposable, we assume $X = (k, \ell)$ and $Y = (a, b)$, then $Y[-1] = (a+1, b+1)$, and $Y[1] = (a - 1, b - 1)$. By the AR-quiver of $A_{n,t}$, both X and Y are overarched by T_1.

$\mathrm{Hom}_{\mathcal{A}_{n,t}}(X, Y) = \mathrm{Ext}^1_{\mathcal{A}_{n,t}}(X, Y[-1]) \simeq D\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[-1], X)$ since $\mathcal{A}_{n,t}$ is 2-CY, and $\mathrm{Hom}_{\mathcal{A}_{n,t}}(Y, X) = \mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y, X[-1]) \simeq \mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[1], X)$. Then we claim that $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[-1], X)$ and $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[1], X)$ cannot be non-zero at the same time. By the correspondence of objects in

$A_{n,t}$ and the diagonals of N-gon P which are $(n+1)$-periodic, and $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y,X) = 0 = \mathrm{Ext}^1_{\mathcal{A}_{n,t}}(X,Y) = 0$, since T is rigid, we have three cases:

(1) Y is overarched by X, then $k \le a < b \le \ell$, see Figure 1. In this case, if $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[-1],X) \neq 0$, then $b=\ell$. Since Y and X are two different objects, this implies a is strictly bigger than k, so $Y[1]$ and X do not cross, that is, $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[1],X) = 0$.

(2) Y is in the counterclockwise direction of X, then $1 \le a < b \le k$, see Figure 2. If $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[-1],X) \neq 0$, then b must equal to k and $a > 1$ since Y and X are two different objects, this implies $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[1],X) = 0$.

(3) Y is in the clockwise direction of X, then $\ell \le a < b \le n+2$, see Figure 3. If $\mathrm{Ext}^1_{\mathcal{A}_{n,t}}(Y[-1],X) \neq 0$, then $b = n+2 = k$ and $a > \ell$. This shows $Y[1]$ does not cross with X.

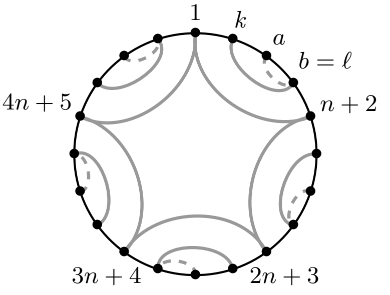

Figure 1 Y is overarched by X

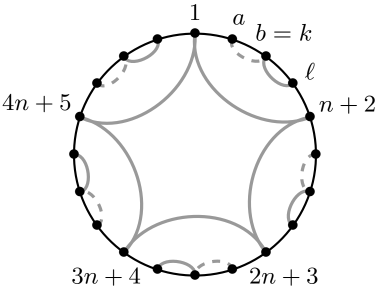

Figure 2 Y is in the counterclockwise direction of X

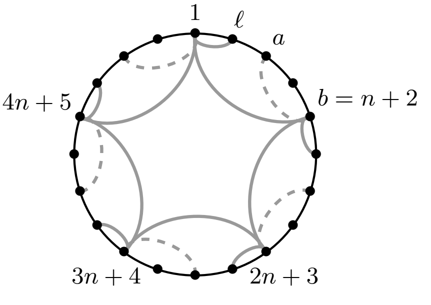

Figure 3 Y is in the clockwise direction of X

3.2 Cluster Structures in $\mathcal{D}_{n,t}$

Similarly, we show the set of maximal rigid objects in $\mathcal{D}_{n,t}$ has a cluster structure.

Let $u = 2t(n+1)$, where $n \geq 1, t \geq 1$, and $\mathcal{C}_{D_u}$ be the cluster category of type D_u. Then $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_u)/\tau^{n+1}\varphi^n$, where φ is induced by an automorphism of D_u of order 2. In the proof of Lemma 2.9 in [6], we know that there exists a covering functor $\pi\colon \mathcal{C}_{D_u} \to \mathcal{D}_{n,t}$.

Write $F := \tau^{(n+1)}\varphi^n$. Then $F : \mathcal{C}_{D_u} \to \mathcal{C}_{D_u}$ can be seen an autoequivalence and $\mathcal{D}_{n,t}$ can be seen as the orbit categories $\mathcal{C}_{D_u}/F$. since $\tau^{-u+1} = [1]$ in $D^b(\mathbb{K}D_u)$ by [20], we have $\tau^{-2t(n+1)} = 1 = \tau^{2t(n+1)}$ in $\mathcal{C}_{\mathcal{D}_\sqcap}$, so π is a $2t$-covering functor and obviously a triangle functor.

For the cluster category $\mathcal{C}_{D_u}$, there is a geometric description [20], let us recall the details for convenience of the readers.

For any $n \geq 1$, we consider a regular $2n$-gon P, we label the vertices of P clockwise by $1, 2, \ldots, 2n$ consecutively. In our arguments below vertices will also be numbered by some $r \in \mathbb{N}$ which might not be in the range $1 \leq r \leq 2n$; in this case the numbering of vertices always has to be taken modulo $2n$.

An *arc* in P is a set $\{i, j\}$ of vertices of P with $j \notin \{i-1, i, i+1\}$, i.e., i and j are different and non-neighboring vertices. The arcs connecting two opposite vertices i and $i+n$ are called *diameters*. We need two different copies of each of these diameters and denote them by $\{i, i+n\}_g$ and $\{i, i+n\}_r$ respectively, where $1 \leq i \leq 2n$. The indices indicate that these diameters are coloured in the colours green and red respectively, which is a convenient way to think about and to visualize the diameters. By a slight abuse of notation, we sometimes omit the indices and just write $\{i, i+n\}$ for diameters, to avoid cumbersome definitions or statements.

Any arc in P which is not a diameter is of the form $\{i, j\}$ where $j \in [i+2, i+n-1]$; here $[i+2, i+n-1]$ stands for the set of vertices of the $2n$-gon P which are met when going clockwise from $i+2$ to $i+n-1$ on the boundary of P. See Figure 4 for an example, for better visibility we draw the red diameters in a wavelike form and the green ones as straight lines.

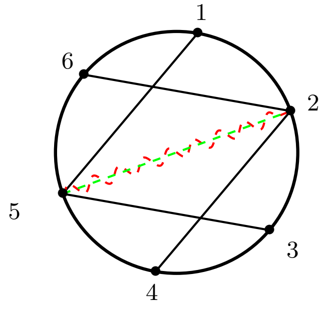

Figure 4 Diameters and non-diameter arcs in 6-gon P

Such an arc has a partner arc $\{i+n, j+n\}$ which is obtained from $\{i, j\}$ by a rotation by 180 degrees. We denote the pair of arcs $\{\{i, j\}, \{i+n, j+n\}\}$ by $\overline{\{i, j\}}$ throughout this paper. The indecomposable objects in $\mathcal{C}_{D_n}$ are in bijection with the union of the set of pairs $\overline{\{i, j\}}$ of non-diameter arcs and the set of diameters $\{i, i+n\}_g$ and $\{i, i+n\}_r$ in two different colours.

This bijection extends to subcategories of $\mathcal{C}_{D_n}$ closed under direct sums and direct summands and collections of arcs of $2n$-gon.

For the pairs of non-diameter arcs $\overline{\{i,j\}}$ the corresponding indecomposable object has coordinates (i,j); note that the coordinates are only determined modulo n so both arcs $\{i,j\}$ and $\{i+n,j+n\}$ in the pair $\overline{\{i,j\}}$ yield the same coordinate in the Auslander–Reiten quiver of $\mathcal{C}_{D_n}$. The action of τ on arcs is rotation by one vertex.

Back to our consideration, for a coordinate (i,j) corresponding to an indecomposable object of $\mathcal{C}_{D_u}$, we denote $[(i,j)]$ by the image under the covering functor π, then $[(i,j)]$ determines an indecomposable object of $\mathcal{D}_{n,t}$. For any subcategory $\mathscr{X}$ in $\mathcal{D}_{n,t}$, the preimage under the covering functor π corresponds to an F-periodic subcategory $\widetilde{\mathscr{X}}=\pi^{-1}(\mathscr{X})$ in $\mathcal{C}_{D_u}$. Moreover, the subcategory $\widetilde{\mathscr{X}}$ corresponds to a set of arcs of $2u$-gon by the discussion above, and we still denote the corresponding set of arcs by $\widetilde{\mathscr{X}}$.

Lemma 3.5 *There are bijections between the following sets:*

(1) *Subcategories $\mathscr{X}$ of $\mathcal{D}_{n,t}$;*

(2) *Collections of arcs $\widetilde{\mathscr{X}}$ of the $2u$-gon P which are F-periodic.*

Definition 3.6 ([14]) (a) *We say that two non-diameter arcs $\{i,j\}$ and $\{k,\ell\}$* cross *precisely if the elements i,j,k,ℓ are all distinct and come in the order i,k,j,ℓ when moving around the $2n$-gon P in one direction or the other (i.e., counterclockwise or clockwise). In particular, the two arcs in $\overline{\{i,j\}}$ do not cross.*

Similarly, in the case $j=i+n$, the above condition defines when a diameter $\{i,i+n\}_g$ (or $\{i,i+n\}_r$) crosses the non-diameter arc $\{k,\ell\}$.

(b) *We say that two pairs $\overline{\{i,j\}}$ and $\overline{\{k,\ell\}}$ of non-diameter arcs* cross *if there exist two arcs in these two pairs which cross in the sense of part* (a). (*Note that then necessarily the other two rotated arcs also cross.*)

Similarly, the diameter $\{i,i+n\}_g$ (or $\{i,i+n\}_r$) crosses the pair $\overline{\{k,\ell\}}$ of non-diameter arcs if it crosses one of the arcs in $\overline{\{k,\ell\}}$. (Note that it then necessarily crosses both arcs in $\overline{\{k,\ell\}}$.)

(c) *Two diameters $\{i,i+n\}_g$ and $\{j,j+n\}_r$ of different colour* cross *if $j\notin\{i,i+n\}$, i.e., if they have different endpoints. But $\{i,i+n\}_g$ and $\{i,i+n\}_r$ do not cross. Moreover, any diameters of the same colour do not cross.*

Suppose X and Y are two indecomposable objects in $\mathcal{C}_{D_n}$, $\overline{\{i,j\}}$ and $\overline{\{k,\ell\}}$ are the corresponding arcs in $2n$-gon P. Then $\mathrm{Ext}_{\mathcal{C}_{D_n}}(X,Y)=0$ if and only if $\overline{\{i,j\}}$ and $\overline{\{k,\ell\}}$ do not cross.

Lemma 3.7 ([6, Lemma 2.9]) (1) *The isomorphism classes of indecomposable rigid objects in $\mathcal{D}_{n,t}$ are parametrised by the arcs $[(i,i+2)],\dots,[(i,i+n+1)]$ for $i=1,\dots,n+1$.*

(2) *The maximal non-crossing F-periodic collections of $4t(n+1)$-gon correspond to isoclasses of basic maximal rigid objects, here $F=\tau^{n+1}\varphi^n$.*

Theorem 3.8 *The set of maximal rigid objects in $\mathcal{D}_{n,t}$ has a cluster structure.*

Proof Note that any diameter in $\mathcal{D}_{n,t}$ is not a rigid object by Lemma 3.7, and condition (a) in Definition 3.6 coincides with the crossing condition in type A, then the proof is similar as Theorem 3.4. □

4 Relationship to Type B Cluster Algebra

In this section, we define the cluster complex $\Delta(\mathcal{A}_{n,t})$ associated to $\mathcal{A}_{n,t}$, and give an isomorphism from $\Delta(\mathcal{A}_{n,t})$ to the cluster complex of root system of B_n, we also do the same thing for $\mathcal{D}_{n,t}$.

A cluster algebra is of finite type if and only if the Cartan counterpart of the principle part of one of its seeds is a cartan matrix of finite type by [11]. Cluster complexes were defined in [12] for finite root systems. They were realized via decorated quiver representations [19], and later via cluster categories of the corresponding quivers [4, 22]. Cluster complex associated to the cluster tube $\mathcal{C}_n$ of rank n (denoted by $\Delta(\mathcal{C}_n)$) was defined in [24], and it was proved that there is an isomorphism between $\Delta(\mathcal{C}_n)$ and the cluster complex of root system of C_{n-1}.

We recall the cluster complex associated to any finite root system from [12]. Let Φ be any finite root system with simple roots $\alpha_1, \ldots, \alpha_n$ and $\Phi_{\geq -1}$ be the set of *almost positive roots* in Φ, that is, the union of positive roots with negative simple roots. According to [12], there exists a unique function

$$\begin{aligned}\Phi_{\geq -1} \times \Phi_{\geq -1} &\to \mathbb{Z}_{\geq 0},\\ (\alpha, \beta) &\longmapsto (\alpha||\beta)\end{aligned}$$

called the *compatibility degree*, by $(\alpha||\beta) = [Y[\alpha] + 1]_{trop}(\beta)$. This notation means evaluating the "tropical specialization" of the Laurent polynomial $Y[\alpha] + 1$. A pair of roots α, β in $\Phi_{\geq -1}$ are *compatible* if $(\alpha||\beta) = 0$ (or equivalently $(\beta||\alpha) = 0$). The *cluster complex* $\Delta(\Phi)$ associated to Φ is a simplicial complex, the set of vertices is $\Phi_{\geq -1}$ and the simplices are mutually compatible subset of $\Phi_{\geq -1}$, the maximal mutually compatible subsets are called *clusters*. This combinatorial object has many interesting properties and applications; we refer the reader to the survey [9] for further reading.

For the categories $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$), we call a set of indecomposable objects a *rigid subset*, provided the direct sum of all indecomposable objects in this set is rigid. Now we define a simplicial complex associated to $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$).

Definition 4.1 *The cluster complex $\Delta(\mathcal{A}_{n,t})$ associated to $\mathcal{A}_{n,t}$ is a simplicial complex whose vertices are the isoclasses of indecomposable rigid objects and whose simplices are the isoclasses of rigid subsets of $\mathcal{A}_{n,t}$. Similarly, we can define the cluster complex $\Delta(\mathcal{D}_{n,t})$ associated to $\mathcal{D}_{n,t}$.*

Let $\mathcal{G}_n$ denote a regular $(2n+2)$-gon. We label the vertices of $\mathcal{G}_n$ clockwise by $1, 2, \ldots, 2n+2$ consecutively. In our arguments below vertices will also be numbered by some $r \in \mathbb{N}$ which might not be in the range $1 \leq r \leq 2n + 2$; in this case the numbering of vertices always has to be taken modulo $2n + 2$.

A *diagonal* in $\mathcal{G}_n$ is a set $[a, b]$ of vertices of $\mathcal{G}_n$ with $b \notin \{a - 1, a, a + 1\}$, i.e., a and b are different and non-neighboring vertices. In particular, the diagonal connecting two opposite vertices a and $a + n + 1$ is called a *diameter*. Thus $[a, b] = [b, a]$, and the centrally symmetric pairs of diagonals are given as $([a,b],[a+n+1,b+n+1])$. Since this pair is uniquely determined by each other, we will sometimes denote the pair by one of its representatives. We shall use the term "cross" to mean "intersect inside the polygon".

Let $\Phi_{\geq -1}$ be the almost positive roots of B_n. By [12], $\Phi_{\geq -1}$ are in bijection with the sets of centrally symmetric pairs of diagonals of $\mathcal{G}_n$, where the diameters are included as degenerate pairs. Under this bijection, compatible sets are collections of mutually non-crossing diagonals. In particular, the clusters correspond to the centrally symmetric triangulations of $\mathcal{G}_n$.

Note that for any indecomposable rigid object $[(a,b)]$ in $\mathcal{A}_{n,t}$, we can always choose the representative (a,b) such that $1 \leq a \leq n+1$. Now, we define a nature map δ, from indecomposable rigid objects in $\mathcal{A}_{n,t}$ to the centrally symmetric pairs of diagonals in $\mathcal{G}_n$ as follows:

$$\delta : (a,b) \mapsto ([a,b],[a+n+1,b+n+1]).$$

Note that under this map, the rigid objects with the same level in the AR-quiver of $\mathcal{A}_{n,t}$ correspond to the diagonals with the same length, and the rigid objects $(i, i+n+1)$ with maximal length, where $1 \leq i \leq n+1$, correspond to the diameters, see Figure 5 for an example.

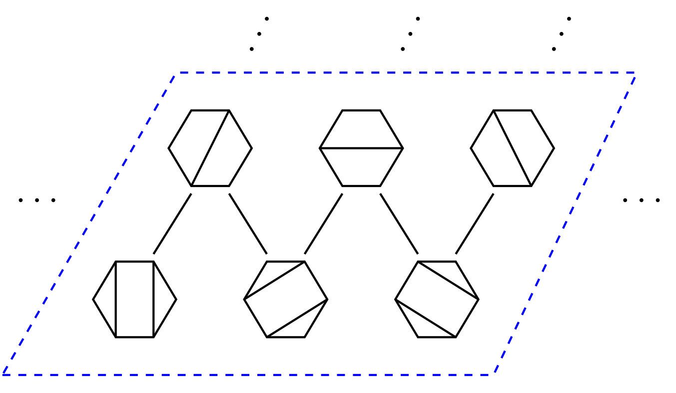

Figure 5 The AR-quiver of $\mathcal{A}_{2,2}$, with the indecomposable rigid objects replaced by their images under the map δ

We denote $\mathrm{Indr}\mathcal{A}_{n,t}$ (resp. $\mathrm{Indr}\mathcal{D}_{n,t}$) the set of indecomposable rigid objects in $\mathcal{A}_{n,t}$ (resp. $\mathcal{D}_{n,t}$).

Lemma 4.2 *The map δ defined above is an isomorphism from* $\mathrm{Indr}\mathcal{A}_{n,t}$ *to the set of centrally symmetric pairs of diagonals of $\mathcal{G}_n$.*

Proof For any indecomposable rigid object (a,b) in $\mathcal{A}_{n,t}$ with $1 \leq a \leq n+1$, since $b-a \leq n+1$, $\delta((a,b)) = ([a,b],[a+n+1,b+n+1])$ is a centrally symmetric pair of diagonals in $\mathcal{G}_n$. Since both $\mathrm{Indr}\mathcal{A}_{n,t}$ and the set of centrally symmetric pairs of diagonals of $\mathcal{G}_n$ are finite sets, it is enough to show δ is epic. For any centrally symmetric pair of diagonals $([a,b],[a+n+1,b+n+1])$ in $\mathcal{G}_n$, we can choose one of its representatives $[a,b]$ such that $1 \leq a \leq n+1$ and $b-a \leq n+1$. This means (a,b) is an indecomposable rigid object in $\mathcal{A}_{n,t}$, that is, $\delta((a,b)) = ([a,b],[a+n+1,b+n+1])$. $\square$

Observation We can see from the above proof that for two indecomposable rigid objects $T_1 = (a,b)$ and $T_2 = (c,d)$ in $A_{n,t}$, $\mathrm{Ext}_{\mathcal{A}_{n,t}}(T_1,T_2) = 0$ if and only if $\delta(a,b)$ and $\delta(c,d)$ do not cross.

Theorem 4.3 *Let Φ^B be the root system of B_n. Then there is an isomorphism from the cluster complex $\Delta(\mathcal{A}_{n,t})$ to the cluster complex $\Delta(\Phi^B)$, which sends vertices to vertices, and simplices to simplices. In particular, the maximal rigid objects correspond to the clusters.*

Proof

$$\begin{tikzcd}
\Phi_{\geq -1} \arrow[drr, "1-1"] & & \\
\mathrm{Indr}\mathcal{A}_{n,t} \arrow[u, dotted] \arrow[r, "\delta"] & \{\text{Centrally symmetric pairs of diagonals of } \mathcal{G}_{\mathrm{n}}\} &
\end{tikzcd}$$

From Lemma 4.2, δ induces an isomorphism from $\mathrm{Indr}\mathcal{A}_{n,t}$ to $\Phi_{\geq -1}$. That is, $\Delta(\mathcal{A}_{n,t}) \simeq \Delta(\Phi^B)$, and sends vertices to vertices. We only need to show that the image under δ of a maximal rigid object coincides with a set of pairs of diagonals which form a centrally symmetric triangulation of $\mathcal{G}_n$. This is clear by Lemma 4.2 and the above observation. $\square$

We have a similar conclusion for $\mathcal{D}_{n,t}$, here we just skip the proof.

Theorem 4.4 *Let Φ^B be the root system of B_n. Then there is an isomorphism from the cluster complex $\Delta(\mathcal{D}_{n,t})$ to the cluster complex $\Delta(\Phi^B)$, which sends vertices to vertices, and simplices to simplices. In particular, the maximal rigid objects correspond to the clusters.*

Let $\mathcal{R}_{\mathcal{A}_{n,t}}$, resp. $\mathcal{R}_{\mathcal{D}_{n,t}}$, be the full subcategory of $\mathcal{A}_{n,t}$, resp. $\mathcal{D}_{n,t}$, generated by the rigid objects. Combing Theorem 4.3 and Theorem 4.4, we have the following corollary.

Corollary 4.5 ([6, Proposition 3.2]) *For all $t \geq 1$, there are equivalences of additive categories*:

(1) $\mathcal{R}_{\mathcal{A}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$,

(2) $\mathcal{R}_{\mathcal{D}_{n,t}} \simeq \mathcal{R}_{\mathcal{A}_{n,1}}$.

Remark 4.6 Let D be a 2-CY triangulated category considered in the paper and let $\mathcal{C}$ be a cluster tube with certain rank. Denote by $\mathcal{R}_{\mathcal{C}}$ (resp. $\mathcal{R}_{\mathcal{D}}$) the full subcategory of $\mathcal{C}$ (resp. $\mathcal{D}$) generated by the rigid objects. By the proof of [6, Proposition 3.2] and the description of the cluster tube in [5], one can check that $\mathcal{C}$ and $\mathcal{D}$ satisfy the assumption of [6, Lemma 3.1]. It follows that there is an isomorphism from $\mathcal{R}_{\mathcal{C}}$ to $\mathcal{R}_{\mathcal{D}}$, which commutes with shifts on objects and irreducible morphisms. Hence $\mathcal{D}$ and $\mathcal{C}$ have the same cluster structure and the same cluster complex. See [5, 6, 24] for details.

Acknowledgements The author would like to express thankfulness to her supervisor, Bin Zhu, for his generous idea and for his different point of view on mathematical idea. The author thanks the referees for their time and comments.

References

[1] Berenstein, A., Fomin, S., Zelevinsky, A.: Cluster algebra III: upper bounds and double Bruhat cells. *Duke Math. J.*, **126**, 1–52 (2005)

[2] Burban, I., Iyama, O., Keller, B., et al.: Cluster tilting for one-dimensional hypersurface singularities. *Adv. Math.*, **217**(6), 2443–2484 (2008)

[3] Buan, A. B., Iyama, O., Reiten, I., et al.: Cluster structures for 2-Calabi–Yau categories and unipotent groups. *Composito Math.*, **145**, 1035–1079 (2009)

[4] Buan, A. B., Marsh, R., Reineke, M., et al.: Tilting theory and cluster combinatorics. *Adv. Math.*, **204**, 572–618 (2006)

[5] Buan, A. B., Marsh, R., Vatne, D.: Cluster structures from 2-Calabi–Yau categories with loops. *Math. Z.*, **265**, 951–970 (2010)

[6] Buan, A. B., Palu, Y., Reiten, I.: Algebras of finite representation type arising from maximal rigid objects. *J. Algebra*, **446**, 426–449 (2016)

[7] Caldero, P., Chapoton, F., Schiffler, R.: Quivers with relations arising from clusters (A_n case). *Tran. Amer. Math. Soc.*, **358**(3), 1347–1364 (2006)

[8] Caldero, P., Keller, B.: From triangulated categories to cluster algebra II. *Ann. Sci. École Norm. Sup.* (4), **39**(6), 938–1009 (2006)

[9] Fomin, S., Reading, N.: Root systems and generalized associahedra. *Math. Comp.*, 63–131 (2005)

[10] Fomin, S., Zelevinsky, A.: Cluster algebra I: foundations. *J. Amer. Math. Soc.*, **15**(2), 497–529 (2002)

[11] Fomin, S., Zelevinsky, A.: Cluster algebra II: Finite type classification. *Invent. Math.*, **154**(1), 63–121 (2003)

[12] Fomin, S., Zelevinsky, A.: Y-systems and generalized associahedra. *Ann. of Math.*, **(**158), 977–1018 (2003)

[13] Fomin, S., Zelevinsky, A.: Cluster algebra IV: Coefficients. *Compositio. Math.*, **143**, 112–164 (2007)

[14] Holm, T., Jørgensen, P., Rubey, M.: Ptolemy diagrams and torsion pairs in the cluster categories of Dynkin type D. *Adv. Appl. Math.*, **51**, 583–605 (2013)

[15] Iyama, O., Yoshino, Y.: Mutations in triangulated categories and rigid Cohen–Macaulay modules. *Invent. Math.*, **172**(1), 117–168 (2008)

[16] Keller, B.: On triangulated orbit categories. *Doc. Math.*, **10**, 551–581 (2005)

[17] Koenig, S., Zhu, B.: From triangulated categories to abelian categories — cluster tilting in a general framework. *Math. Z.*, **258**(1), 143–160 (2008)

[18] Marsh, R.: Lecture notes on cluster algebras. European Math. Society, Zurich, 2013

[19] Marsh, R., Reineke, M. Zelevinsky, A.: Generalized associahedra via quiver representations. *Trans. Amer. Math. Soc.*, **355**(10), 4171–4186 (2003)

[20] Schiffler, R.: A geometric model for cluster categories of type D_n. *J. Algebr. Comb.*, **27**, 1–21 (2008)

[21] Xu, J., Ouyang, B.: Maximal rigid objects without loops in connected 2-CY categories are cluster tilting objects. *J. Algebra Appl.*, **14**(5), 13 pages (2015)

[22] Zhu, B.: BGP-reflection functors and cluster combinatorics. *J. Pure Appl. Algebra*, **209**, 497–506 (2007)

[23] Zhou, Y., Zhu, B.: Maximal rigid subcategories in 2-Calabi–Yau triangulated categories. *J. Algebra*, **348**, 49–60 (2011)

[24] Zhou, Y., Zhu, B.: Cluster algebras arising from cluster tubes. *J. London Math. Soc.*, **89**(2), 703–723 (2014)