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# Smoothness of the Gradient of Weak Solutions of Degenerate Linear Equations

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**Abstract** Let Q(x) be a nonnegative definite, symmetric matrix such that  $\sqrt{Q(x)}$  is Lipschitz continuous. Given a real-valued function b(x) and a weak solution u(x) of  $\operatorname{div}(Q\nabla u) = b$ , we find sufficient conditions in order that  $\sqrt{Q}\nabla u$  has some first order smoothness. Specifically, if  $\Omega$  is a bounded open set in  $\mathbb{R}^n$ , we study when the components of  $\sqrt{Q}\nabla u$  belong to the first order Sobolev space  $W_Q^{1,2}(\Omega)$ defined by Sawyer and Wheeden. Alternately, we study when each of n first order Lipschitz vector field derivatives  $X_i u$  has some first order smoothness if u is a weak solution in  $\Omega$  of  $\sum_{i=1}^n X'_i X_i u + b = 0$ . We do not assume that  $\{X_i\}$  is a Hörmander collection of vector fields in  $\Omega$ . The results signal ones for more general equations.

**Keywords** Degenerate elliptic differential equations, degenerate quadratic forms, weak solutions, second order regularity

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## 1 Introduction

We begin with some notation and background. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and Q(x) be a nonnegative definite, symmetric  $n \times n$  matrix on  $\Omega$ . Assume that  $\sqrt{Q(x)}$  is Lipschitz continuous in  $\Omega$ , i.e., that the entries of  $\sqrt{Q(x)}$  are Lipschitz continuous in  $\Omega$ , and let

$$Q(x,\xi) = \xi \cdot Q(x)\xi = \sqrt{Q(x)}\xi \cdot \sqrt{Q(x)}\xi, \quad x \in \Omega, \xi \in \mathbb{R}^n,$$
(1.1)

denote the quadratic form corresponding to Q. Note that if Q is singular at a point x, then  $Q(x,\xi)$  vanishes at some  $\xi \neq 0$ . Following [6], we consider the Sobolev space  $W_Q^{1,2}(\Omega)$  of all pairs  $(u(x), \mathbf{v}(x)), x \in \Omega$ , where u is real-valued and  $\mathbf{v}$  is a vector in  $\mathbb{R}^n$ , such that there is a sequence  $\{u_k(x)\}_{k=1}^{\infty}$  of Lipschitz functions on  $\Omega$  satisfying

$$\lim_{k \to \infty} \left\{ \|u_k - u\|_{L^2(\Omega)} + \left( \int_{\Omega} Q(x, \nabla u_k(x) - \mathbf{v}(x)) \, dx \right)^{\frac{1}{2}} \right\} = 0.$$
(1.2)

Equivalently, as  $k \to \infty$ ,

 $u_k \to u \text{ in } L^2(\Omega) \quad \text{and} \quad \sqrt{Q} \nabla u_k \to \sqrt{Q} \mathbf{v} \text{ in } L^2(\Omega).$ 

Later we will see that the functions  $u_k$  in the approximating sequence  $\{u_k\}$  can be chosen to belong to  $\operatorname{Lip}(\Omega) \cap C^{\infty}(\Omega)$ .

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If  $(u, \mathbf{v}) \in W_Q^{1,2}(\Omega)$ , then by [6], since  $\sqrt{Q} \in \operatorname{Lip}(\Omega)$ , the vector  $\mathbf{v}$  is uniquely determined by the first entry u in the sense that if two pairs  $(u, \mathbf{v}_1), (u, \mathbf{v}_2)$  with first coordinate u belong to  $W_Q^{1,2}(\Omega)$ , then  $\sqrt{Q}\mathbf{v}_1 = \sqrt{Q}\mathbf{v}_2$  a.e. in  $\Omega$ . However,  $\mathbf{v}_1$  may differ from  $\mathbf{v}_2$  at points where  $\sqrt{Q}$  is singular. In some cases when  $\sqrt{Q} \notin \operatorname{Lip}(\Omega)$ , uniqueness of the second entry fails dramatically; see e.g. [6]. If  $(u, \mathbf{v}) \in W_Q^{1,2}(\Omega)$ , we will denote  $\mathbf{v} = \nabla u$  although u may not have an ordinary weak gradient in  $\Omega$ . We often write  $u \in W_Q^{1,2}(\Omega)$  instead of  $(u, \nabla u) \in W_Q^{1,2}(\Omega)$ , and we set

$$\|u\|_{W^{1,2}_Q(\Omega)} = \left(\int_{\Omega} |u(x)|^2 dx + \int_{\Omega} |\sqrt{Q(x)}\nabla u(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{if } u \in W^{1,2}_Q(\Omega).$$
(1.3)

Let  $b \in L^1_{loc}(\Omega)$ . Suppose  $u \in W^{1,2}_Q(\Omega)$  and u is a weak solution in  $\Omega$  of the equation

$$\operatorname{div}(Q(x)\nabla u(x)) = b(x), \quad x \in \Omega.$$
(1.4)

Here we say as usual that u is a weak solution in  $\Omega$  of (1.4) if

$$\int_{\Omega} (Q(x)\nabla u(x)) \cdot \nabla \Phi(x) \, dx + \int_{\Omega} b(x)\Phi(x) \, dx = 0 \quad \text{for all } \Phi \in \operatorname{Lip}_0(\Omega).$$
(1.5)

The main purpose of this paper is to find conditions implying that the components of  $\sqrt{Q}\nabla u$ then belong to  $W_Q^{1,2}(\Omega)$ . We will consistently denote these components by  $V_i \cdot \nabla u$ ,  $i = 1, \ldots, n$ , where  $V_i(x)$  is the *i*-th row vector of  $\sqrt{Q(x)}$ . Our main result is as follows.

**Theorem 1.1** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and Q(x) be a nonnegative definite, symmetric  $n \times n$  matrix on  $\Omega$  satisfying  $\sqrt{Q} \in C^{1,1}(\Omega)$ . Suppose that  $u, b \in W_Q^{1,2}(\Omega)$  and u is a weak solution in  $\Omega$  of (1.4) represented in  $W_Q^{1,2}(\Omega)$  by a sequence  $\{u_k\}_{k=1}^{\infty}$  of smooth functions. Let  $\{w_\ell\}_\ell$  be the entries of  $\sqrt{Q}$  and let  $\{S_k\}$  be the sequence of numbers defined by

$$S_k = \left(\int_{\Omega} \left| \left\{ \sum_{\ell} |w_{\ell}(x)| \right\} \nabla u_k(x) \right|^2 dx \right)^{\frac{1}{2}} = \left\| \sum_{\ell} |w_{\ell}| \nabla u_k \right\|_{L^2(\Omega)}.$$
 (1.6)

If

$$\limsup_{k \to \infty} S_k < \infty, \tag{1.7}$$

then every component  $V_i \cdot \nabla u$  of  $\sqrt{Q} \nabla u$  belongs to  $W^{1,2}_Q(\Omega)$  and

$$\|V_i \cdot \nabla u\|_{W^{1,2}_Q(\Omega)} \le C \limsup_{k \to \infty} S_k, \tag{1.8}$$

where C is a constant that depends only on  $\|b\|_{W^{1,2}_Q(\Omega)}$  and the sum of  $\|w_\ell\|_{L^{\infty}(\Omega)}$  and the Lipschitz constants in  $\Omega$  of the  $w_\ell$  and their first partial derivatives.

We now list some comments about condition (1.7). First, if  $u \in W_Q^{1,2}(\Omega)$  and  $\{u_k\}$  represents u in  $W_Q^{1,2}(\Omega)$ , then  $\|\sqrt{Q}\nabla u_k - \sqrt{Q}\nabla u\|_{L^2(\Omega)} \to 0$  as  $k \to \infty$ , or equivalently,  $\|V_i \cdot \nabla u_k - V_i \cdot \nabla u\|_{L^2(\Omega)} \to 0$  for every row vector  $V_i$  of  $\sqrt{Q}$ . Hence  $\|V_i \cdot \nabla u_k\|_{L^2(\Omega)}$  is bounded in k for every i. On the other hand, assumption (1.7) is much stronger since

$$|V_i \cdot \nabla u_k| \le \left|\sum_{\ell} |w_{\ell}| \nabla u_k\right|$$
 for every *i* and *k*.

Next, (1.7) can be rewritten in vector field notation as

$$\limsup_{k \to \infty} \left\| \left( \sum_{\ell} |w_{\ell}| \right) e_i \cdot \nabla u_k \right\|_{L^2(\Omega)} < \infty, \quad i = 1, \dots, n,$$

where  $e_i$  is the *i*-th unit vector. This clearly holds if *u* belongs to the Sobolev space  $W_{Q^*}^{1,2}(\Omega)$ , where  $Q^*$  is the (diagonal) matrix with rows  $(\sum_{\ell} |w_{\ell}|)^2 e_i$ . For example, since every  $w_{\ell}$  is bounded in  $\Omega$ , it holds if *u* belongs to the classical Sobolev space  $W^{1,2}(\Omega)$  since then the  $u_k$  can be chosen to satisfy  $\|\nabla u_k - \nabla u\|_{L^2(\Omega)} \to 0$ . Furthermore, in case  $\sum_{\ell} |w_{\ell}|$  is bounded away from 0 in  $\Omega$ , then  $S_k \geq c \|\nabla u_k\|_{L^2(\Omega)}$  for some positive constant *c* independent of *k*, and then for any sequence  $\{u_k\}$  of Lipschitz functions, (1.7) is equivalent to  $\limsup_{k\to\infty} \|\nabla u_k\|_{L^2(\Omega)} < \infty$ . Finally, we do not know if the conclusion of Theorem 1.1 remains true if (1.7) is replaced by finiteness of some number smaller than  $\limsup_{k\to\infty} S_k$  and all other hypotheses of the theorem are retained.

Under our assumptions that  $\Omega$  is bounded and  $\sqrt{Q}$  is Lipschitz continuous in  $\Omega$ ,  $W_Q^{1,2}(\Omega)$ can be identified with a well-known class of functions having weak vector field derivatives, and Theorem 1.1 can be restated in terms of this class, as we now discuss. Let V(x) be a Lipschitz vector in  $\Omega$ , and let  $X = V \cdot \nabla$  be the corresponding vector field derivative. A function  $g \in L^1_{\text{loc}}(\Omega)$  is called the weak derivative Xu in  $\Omega$  of a function  $u \in L^2_{\text{loc}}(\Omega)$  if for all  $\varphi \in \text{Lip}_0(\Omega)$ ,

$$\int_{\Omega} g \varphi \, dx = -\int_{\Omega} u \, X' \varphi \, dx = -\int_{\Omega} u \operatorname{div}(\varphi V) \, dx$$
$$= -\int_{\Omega} u \left\{ V \cdot \nabla \varphi + \varphi \operatorname{div} V \right\} dx. \tag{1.9}$$

The weak derivative Xu is clearly unique if it exists. If  $u \in W_Q^{1,2}(\Omega)$  and  $\sqrt{Q} \in \operatorname{Lip}(\Omega)$ , then a simple limit argument based on integration by parts shows that if  $X_i = V_i \cdot \nabla$ , where  $\{V_i\}_{i=1}^n$  are the row vectors of  $\sqrt{Q}$ , then each weak derivative  $X_i u$  exists in  $\Omega$  and  $X_i u = V_i \cdot \nabla u \in L^2(\Omega)$ . Denoting  $\mathcal{X} = \{X_i\}_{i=1}^n$ , the collection  $H_{\mathcal{X}}^{1,2}(\Omega)$  of all  $u \in L^2(\Omega)$  such that every  $X_i u \in L^2(\Omega)$ thus contains  $W_Q^{1,2}(\Omega)$ . By [2, 3] and [6], the converse is also true, namely, if the rows  $V_i$  of  $\sqrt{Q}$  are Lipschitz vectors in  $\Omega$ , then  $H_{\mathcal{X}}^{1,2}(\Omega) \subset W_Q^{1,2}(\Omega)$ . See [6] for some other conditions that guarantee this equivalence.

As a consequence, Theorem 1.1 can be rephrased as follows in terms of a given collection  $\mathcal{X} = \{X_i = V_i \cdot \nabla\}_{i=1}^n$  of vector field derivatives in  $\Omega$ . Let each  $V_i \in C^{1,1}(\Omega)$  and suppose that  $u, b \in H^{1,2}_{\mathcal{X}}(\Omega)$  and u is a weak solution in  $\Omega$  of  $\sum_i X'_i X_i u + b = 0$ . Choose a sequence  $\{u_k\}$  of smooth functions satisfying

$$\lim_{k \to \infty} \left\{ \|u_k - u\|_{L^2(\Omega)} + \|X_i u_k - X_i u\|_{L^2(\Omega)} \right\} = 0 \quad \text{for every } i.$$

If  $\limsup_{k\to\infty} \left\| \sum_{\ell} |V_{\ell}| \nabla u_k \right\|_{L^2(\Omega)} < \infty$ , then every  $X_i u \in H^{1,2}_{\mathcal{X}}(\Omega)$ , and for  $j = 1, \ldots, n$ ,

$$\|X_j X_i u\|_{L^2(\Omega)} \le C \limsup_{k \to \infty} \left\| \sum_{\ell} |V_{\ell}| \nabla u_k \right\|_{L^2(\Omega)}$$

with C as in Theorem 1.1.

### 2 Preliminaries

The proof of Theorem 1.1 uses relations between  $W_Q^{1,2}(\Omega)$  and difference quotients along integral curves associated with row vectors of  $\sqrt{Q}$ . These relations are derived in this section. See for example [4, Section 7.11] for analogues in the case of ordinary  $W^{1,2}(\Omega)$  and ordinary difference quotients. We begin by recalling some things from Introduction. Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $W_Q^{1,2}(\Omega)$  be the Sobolev space associated as above with a nonnegative definite  $n \times n$  symmetric matrix Q = Q(x),  $x \in \Omega$ . We assume that the entries of  $\sqrt{Q}$  belong to the class  $\operatorname{Lip}(\Omega)$  of Lipschitz continuous functions in  $\Omega$ , and often that they also belong to  $C^1(\Omega)$  or even to  $C^{1,1}(\Omega)$ . For each  $i = 1, \ldots, n$ , let  $V_i(x)$  be the vector equal to the *i*-th row of  $\sqrt{Q(x)}$ , and let  $X_i(x) = V_i(x) \cdot \nabla$  be the corresponding vector field derivative. Denote the collection of these vector fields by  $\mathcal{X}(x) = \{X_i(x)\}_{i=1}^n = \{V_1(x) \cdot \nabla, \ldots, V_n(x) \cdot \nabla\}, x \in \Omega$ . Since the entries of  $\sqrt{Q}$  belong to  $\operatorname{Lip}(\Omega)$ , the same is true for the entries of each  $V_i$ . Then  $W_Q^{1,2}(\Omega)$  coincides with the Sobolev space  $H^{1,2}_{\mathcal{X}}(\Omega)$  of functions  $w \in L^2(\Omega)$  with weak derivatives  $X_i w$  in  $L^2(\Omega)$  (see [2, 3, 6]), where a function  $\mu \in L^1_{\operatorname{loc}}(\Omega)$  is called the weak derivative  $X_i w$  of w if for all  $\varphi$  in the class  $\operatorname{Lip}_0(\Omega)$  of functions in  $\operatorname{Lip}(\Omega)$  with compact support in  $\Omega$ ,

$$\int_{\Omega} \mu \varphi \, dx = -\int_{\Omega} w \, X'_i \varphi \, dx = -\int_{\Omega} w \operatorname{div}(\varphi V_i) \, dx$$
$$= -\int_{\Omega} w \left( V_i \cdot \nabla \varphi + \varphi \operatorname{div} V_i \right) dx.$$

Each weak derivative  $X_i w$  is uniquely determined up to a set of Lebesgue measure zero. Furthermore, if  $Q(x,\xi)$  is the quadratic form defined by (1.1), then since

$$Q(x,\xi) = \sum_{i=1}^{n} (V_i(x) \cdot \xi)^2,$$

(1.2) implies that if  $w \in W^{1,2}_Q(\Omega)$ , there is a sequence  $\{w_k\}_{k=1}^{\infty} \subset \operatorname{Lip}(\Omega)$  such that

$$\|w - w_k\|_{L^2(\Omega)} + \sum_{i=1}^n \|X_i w - V_i \cdot \nabla w_k\|_{L^2(\Omega)} \to 0 \quad \text{as } k \to \infty.$$
 (2.1)

We denote  $X_i w = V_i \cdot \nabla w$  and  $\sqrt{Q} \nabla w = (X_1 w, \dots X_n w)$ .

The approximating functions in (2.1) can be chosen to satisfy  $w_k \in C^{\infty}(\Omega) \cap \operatorname{Lip}(\Omega)$  since  $\Omega$ is bounded and  $\sqrt{Q}$  is bounded in  $\Omega$ . To see why, first extend each  $w_k \in \operatorname{Lip}(\Omega)$  to a Lipschitz function on all of  $\mathbb{R}^n$  (see e.g. [7, Theorem 7.64]) and multiply the extension by a smooth function with compact support in  $\mathbb{R}^n$  equal to 1 on  $\overline{\Omega}$ . By convolving the result with a smooth compactly supported approximation of the identity, it follows from standard facts that there are functions  $\tilde{w}_k \in C^{\infty}(\Omega) \cap \operatorname{Lip}(\Omega)$  such that  $w_k - \tilde{w}_k \to 0$  in the norm of the classical Sobolev space  $W^{1,2}(\Omega)$ . Then, for every *i*, since  $V_i$  is bounded in  $\Omega$ ,

$$\|V_i \cdot (\nabla w_k - \nabla \tilde{w}_k)\|_{L^2(\Omega)} \le \|V_i\|_{L^\infty(\Omega)} \|\nabla w_k - \nabla \tilde{w}_k\|_{L^2(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

Now let V = V(x) be a Lipschitz vector field in  $\Omega$ . For  $t \in \mathbb{R}$ , consider the integral curves  $\gamma(t, x) = \gamma(V, t, x)$  given by

$$\gamma'(t,x) = V(\gamma(t,x)), \quad \gamma(0,x) = x.$$
(2.2)

Here  $\gamma'(t, x)$  denotes the *t*-derivative of  $\gamma(t, x)$ . If  $x \in \Omega$  and  $\delta(x) = \text{dist}_{\text{euc}}(x, \partial\Omega)$ , then according to Picard's theorem, there is a positive constant  $c_V$  depending only on  $\|V\|_{L^{\infty}(\Omega)}$  and the Lipschitz constant of V (denoted  $\|V\|_{\text{Lip}(\Omega)}$ ) such that  $\gamma(t, x)$  exists, lies in  $\Omega$ , and is unique if  $|t| < c_V \delta(x)$ . In what follows,  $c_V$  and  $C_V$  will denote various constants depending only on  $\|V\|_{L^{\infty}(\Omega)}$  and  $\|V\|_{\text{Lip}(\Omega)}$ . Let us recall some facts about integral curves. For t and x as above,  $\gamma(t, x)$  satisfies the integral equation

$$\gamma(t,x) = x + \int_0^t V(\gamma(\tau,x)) \, d\tau, \qquad (2.3)$$

and consequently

$$|\gamma(t,x) - x| = \left| \int_0^t V(\gamma(\tau,x)) \, d\tau \right| \le C_V |t|.$$

Also, if  $x \in \Omega$  and  $t, s \in \mathbb{R}$  satisfy  $|t|, |s| \leq c_V \delta(x)$ , then

$$|\gamma(t,x) - \gamma(s,x)| = \left| \int_s^t V(\gamma(\tau,x)) \, d\tau \right| \le C_V |t-s|.$$

Therefore  $\gamma(t, x)$  is Lipschitz continuous in t if  $|t| < c_V \delta(x)$ , and its Lipschitz constant is independent of x. See also Lemma 4.1.

It is easy to show that  $\gamma(t, x)$  is also locally Lipschitz continuous in x uniformly in t for small t. In fact, if  $x, y \in \Omega$  and  $|t| < c_V \min{\{\delta(x), \delta(y)\}}$ , then

$$\begin{aligned} |\gamma(t,x) - \gamma(t,y)| &= \left| (x-y) + \int_0^t \left[ V(\gamma(\tau,x)) - V(\gamma(\tau,y)) \right] d\tau \\ &\leq |x-y| + C_V|t| \sup_{\tau: |\tau| \leq |t|} |\gamma(\tau,x) - \gamma(\tau,y)|, \end{aligned}$$

and hence, for all  $\theta$  with  $0 < \theta < c_V \min{\{\delta(x), \delta(y)\}}$ , we have

$$\sup_{t:|t|\leq\theta} |\gamma(t,x) - \gamma(t,y)| \leq |x-y| + C_V \theta \sup_{t:|t|\leq\theta} |\gamma(t,x) - \gamma(t,y)|$$

Now choosing  $\theta$  to also satisfy  $C_V \theta \leq 1/2$  and subtracting the second term on the right from both sides, we obtain

$$\sup_{t:|t|\le\theta} |\gamma(t,x) - \gamma(t,y)| \le 2|x-y|, \tag{2.4}$$

as claimed.

In particular, these estimates imply that for every compact set  $K \subset \Omega$ , there is a positive constant  $\delta_{V,K}$  depending only on V and K such that  $\gamma(t,x)$  is Lipschitz continuous in (t,x) if  $|t| < \delta_{V,K}$  and  $x \in K$ , with Lipschitz constant independent of K.

For the integral curves  $\gamma(t, x) = \gamma(V, t, x)$ , let us now consider the Jacobian  $\partial z/\partial x$  of the change of variables from x to z given by

$$z = \gamma(t, x)$$
 for fixed t.

We always assume that (t, x) satisfies  $x \in \Omega$  and  $|t| < c_V \delta(x)$ , but unless we make a further restriction or convention, there may be no fixed value of t such that  $\gamma(t, x)$  is defined for all  $x \in \Omega$ , in particular for those x near  $\partial \Omega$ . For example, in the next two lemmas, x will lie in an open set  $\Omega' \subset \Omega$  with compact closure in  $\Omega$ , and then there is a constant  $\delta > 0$  depending on  $\Omega'$ and V such that  $\gamma(t, x)$  is defined for all (t, x) with  $|t| < \delta$  and  $x \in \Omega'$ . See also the convention used in Section 3 for a similar purpose.

To compute the Jacobian, we will assume that  $V \in \text{Lip}(\Omega) \cap C^1(\Omega)$ . This guarantees that the chain rule can be applied to  $V(\gamma(t, x))$  as a function of x, and also by [1, 5] that  $\gamma(t, x)$  is continuously differentiable in x. Denoting  $V(y) = (v_1(y), \ldots, v_n(y))$  and  $x = (x_1, \ldots, x_n)$ , let us show that

$$\det \frac{\partial z}{\partial x} = 1 + \int_0^t \left[ \sum_{k=1}^n \left( \frac{\partial}{\partial x_k} \gamma(s, x) \right) \cdot (\nabla v_k) (\gamma(s, x)) \right] ds + R(t, x),$$
(2.5)

where the remainder R(t, x) satisfies the following estimate: there are constants  $c_V, C_V > 0$ such that

$$|R(t,x)| \le C_V t^2 \quad \text{if } x \in \Omega \text{ and } |t| < c_V \delta(x).$$
(2.6)

Both (2.5) and (2.6) can be deduced from a formula for the determinant of the Jacobian given for example in [1]. Alternately, they can be verified by straightforward computation as follows. Denote  $z = \gamma(t, x) = (\gamma_1(t, x), \ldots, \gamma_n(t, x))$ . Then

$$\det \frac{\partial z}{\partial x} = \det \left( \frac{\partial}{\partial x_j} \gamma_i(t, x) \right)_{ij} \quad \text{and}$$
$$\gamma'_i(t, x) = v_i(\gamma(t, x)), \quad \text{so that} \quad \gamma_i(t, x) = x_i + \int_0^t v_i(\gamma(s, x)) ds.$$

Therefore,

$$\frac{\partial}{\partial x_j} \gamma_i(t, x) = \delta_{ij} + \int_0^t \left(\frac{\partial}{\partial x_j} \gamma(s, x)\right) \cdot (\nabla v_i)(\gamma(s, x)) \, ds \quad \text{and} \\
\frac{\partial z}{\partial x} = I + D(t, x) \quad \text{with} \\
D(t, x) = \left(\int_0^t \left(\frac{\partial}{\partial x_j} \gamma(s, x)\right) \cdot (\nabla v_i)(\gamma(s, x)) \, ds\right)_{ij}.$$
(2.7)

Formula (2.5) now follows by expanding the resulting determinant, which has the form det  $\frac{\partial z}{\partial x} = \det(I + D(t, x))$ . In fact,  $\det(I + D(t, x))$  is the characteristic polynomial  $P(\lambda)$  of D(t, x) evaluated at  $\lambda = -1$ . Thus,  $P(-1) = \sum_{k=0}^{n} M_k$  where  $M_0 = 1$  and  $M_k$  for  $k \ge 1$  is the sum of the determinants of the  $k \times k$  principal minors of D(t, x). Since

$$M_1 = \text{trace } D(t, x) = \int_0^t \left[ \sum_{k=1}^n \left( \frac{\partial}{\partial x_k} \gamma(s, x) \right) \cdot (\nabla v_k)(\gamma(s, x)) \right] ds$$

it follows that (2.5) holds with  $R(t,x) = \sum_{k=2}^{n} M_k$ . The size estimate (2.6) is then immediate since every entry of D(t,x) is bounded in absolute value by  $C_V|t|$  due to the Lipschitz behaviors of  $\gamma(x,t)$  and V.

Similarly,

$$|\text{trace } D(t,x)| \le C_V |t| \text{ if } x \in \Omega \text{ and } |t| < c_V \delta(x).$$

Since trace D(t, x) is the same as the integral term in (2.5), (2.5) can be rewritten as

$$\det \frac{\partial z}{\partial x} = 1 + \text{trace } D(t, x) + R(t, x)$$

and hence

$$\left| \det \frac{\partial z}{\partial x} - 1 \right| \le C_V |t| \quad \text{if } x \in \Omega \text{ and } |t| < c_V \delta(x).$$
 (2.8)

We will often abbreviate (2.8) by writing det  $\partial z/\partial x = 1 + O(t)$ .

The next two lemmas show how difference quotients of the form  $[u(\gamma(h, x)) - u(x)]/h$  are related to the vector field derivative  $(V \cdot \nabla u)(x)$ .

**Lemma 2.1** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ , Q(x) be a nonnegative definite matrix and V(x) be a vector field with  $(V(x) \cdot \xi)^2 \leq Q(x,\xi)$  if  $x \in \Omega, \xi \in \mathbb{R}^n$ . Assume that  $V \in \text{Lip}(\Omega) \cap C^1(\Omega)$  and let  $\gamma(t,x) = \gamma(V,t,x)$  satisfy (2.2). If  $\Omega' \subset \Omega$  is an open set with compact closure in  $\Omega$ , there is a positive constant  $\delta$  depending on  $\Omega'$  and V such that

$$\sup_{h:0<|h|<\delta} \left(\int_{\Omega'} \left|\frac{u(\gamma(h,x)) - u(x)}{h}\right|^2 dx\right)^{\frac{1}{2}} \le 2\left(\int_{\Omega} |(V \cdot \nabla u)(x)|^2 dx\right)^{\frac{1}{2}} \le 2\left(\int_{\Omega} \left|\left(\sqrt{Q}\nabla u\right)(x)\right|^2 dx\right)^{\frac{1}{2}} \le 2\left(\int_{\Omega} \left|\left(\sqrt{Q}\nabla u\right)(x)\right|^2 dx\right)^{\frac{1}{2}}$$
(2.9)

for any  $u \in W^{1,2}_Q(\Omega)$ .

*Proof* Fix  $\Omega'$  as in the hypothesis and choose  $\delta > 0$  depending on  $\Omega'$  and V such that  $\gamma(h, x) \subset \Omega$  if  $x \in \Omega'$  and  $|h| < \delta$ . Assuming first that  $u \in C^1(\Omega)$ , we have

$$\frac{u(\gamma(h,x)) - u(x)}{h} = \frac{1}{h} \int_0^h \frac{d}{dt} \left[ u(\gamma(t,x)) \right] dt$$
$$= \frac{1}{h} \int_0^h (\nabla u)(\gamma(t,x)) \cdot \gamma'(t,x) dt$$
$$= \frac{1}{h} \int_0^h (\nabla u)(\gamma(t,x)) \cdot V(\gamma(t,x)) dt$$

Therefore,

$$\left|\frac{u(\gamma(h,x)) - u(x)}{h}\right|^2 \le \frac{1}{h} \int_0^h |(V \cdot \nabla u)(\gamma(t,x))|^2 dt.$$

Integrating with respect to x over  $\Omega'$  gives

$$\int_{\Omega'} \left| \frac{u(\gamma(h,x)) - u(x)}{h} \right|^2 dx \le \frac{1}{h} \int_0^h \int_{\Omega'} |(V \cdot \nabla u)(\gamma(t,x))|^2 dx \, dt.$$

In the inner integral on the right side, with t fixed, we make the change of variables  $z = \gamma(t, x)$ . Using (2.8), it follows by choosing  $\delta$  smaller if necessary that  $|\det(\partial z/\partial x)| \ge 1/2$  if  $|t| < \delta$  and  $x \in \Omega'$ . The integral on the right is then at most

$$2\frac{1}{h}\int_0^h \int_\Omega |(V\cdot\nabla u)(z)|^2 dz \, dt = 2\int_\Omega |(V\cdot\nabla u)(z)|^2 dz.$$

This proves the lemma when  $u \in C^1(\Omega)$ . The general case follows by approximation using (2.1) and the comments in the paragraph after (2.1); note that if  $\{u_k(x)\}$  converges in  $L^2(\Omega)$  to u(x)and K is a compact set in  $\Omega$ , then  $\{u_k(\gamma(h, x))\}$  converges in  $L^2(K)$  to  $u(\gamma(h, x))$  for all fixed small h by a similar change of variables argument. Furthermore,  $V \cdot \nabla u_k \to V \cdot \nabla u$  in  $L^2(\Omega)$ if  $u_k \to u$  in  $W^{1,2}_Q(\Omega)$ , and we have  $\|V \cdot \nabla u\|_{L^2(\Omega)} \leq \|\sqrt{Q}\nabla u\|_{L^2(\Omega)}$  by hypothesis, completing the proof.

The integral curves  $\gamma(t, x) = \gamma(V, t, x)$  have a well-known translation property, namely, there is a constant  $c_V > 0$  such that

$$\gamma(t,x) = \gamma(t-h,\gamma(h,x)) \quad \text{if } |t|, |h| < c_V \delta(x) \text{ and } x \in \Omega.$$
(2.10)

In fact, as functions of t, both  $\gamma(t, x)$  and  $\gamma(t - h, \gamma(h, x))$  are integral curves of the same Lipschitz vector field V, and they are equal at t = h since

$$\gamma(t-h,\gamma(h,x))\Big|_{t=h} = \gamma(0,\gamma(h,x)) = \gamma(h,x).$$

Hence, (2.10) is a corollary of the uniqueness of integral curves once we show that  $\gamma(t-h, \gamma(h, x))$  exists for x, t, h as in (2.10) with a suitable constant  $c_V$ . We know there are constants c, C > 0 depending only on V such that  $\gamma(t, x) \subset \Omega$  and  $|\gamma(t, x) - x| < C|t|$  if  $x \in \Omega$  and  $|t| < c\delta(x)$ . Then if  $x \in \Omega$  and  $|h| < c\delta(x)$ ,

$$\begin{split} \delta(\gamma(h,x)) &\geq \delta(x) - |\gamma(h,x) - x| \\ &> \delta(x) - C|h| \\ &> \frac{1}{2}\delta(x) \quad \text{if } |h| < \delta(x)/(2C). \end{split}$$

If also  $|t|, |h| < (c/4)\delta(x)$ , then

$$|t - h| \le |t| + |h| < (c/2)\delta(x) < c\delta(\gamma(h, x)).$$

It follows that (2.10) holds for some  $c_V > 0$ .

By choosing t = 0 in (2.10), we obtain a formula for the inverse of the mapping  $x \to z = \gamma(h, x)$ , namely,

$$x = \gamma(-h, \gamma(x, h))$$
 if  $x \in \Omega$  and  $|h| < c_V \delta(x)$ . (2.11)

**Lemma 2.2** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ , V(x) be a vector in  $\operatorname{Lip}(\Omega) \cap C^1(\Omega)$ , and  $\gamma(h, x) = \gamma(V, h, x)$  satisfy (2.2). Let  $\Omega'$  be an open set with compact closure in  $\Omega$ . If  $u \in L^2(\Omega)$  and for some  $\delta > 0$  and some finite L,

$$\sup_{h:0<|h|<\delta} \int_{\Omega'} \left| \frac{u(\gamma(h,x)) - u(x)}{h} \right|^2 dx \le L^2,$$
(2.12)

then there is a sequence  $\{h_k\} \to 0$  such that  $[u(\gamma(h_k, x)) - u(x)]/h_k$  converges weakly in  $L^2(\Omega')$ to a function w(x). Furthermore,  $||w||_{L^2(\Omega')} \leq L$ , the vector field derivative  $V \cdot \nabla u$  exists in the weak sense in  $\Omega'$ , and  $V \cdot \nabla u = w$  in  $\Omega'$ . In case (2.12) holds for every open set  $\Omega'$  that has compact closure in  $\Omega$  (for some  $\delta$  depending on  $\Omega'$ ) with L independent of  $\Omega'$ , then  $V \cdot \nabla u$ exists in the weak sense in  $\Omega$  and  $||V \cdot \nabla u||_{L^2(\Omega)} \leq L$ .

**Proof** Fix  $\Omega'$  and let u satisfy (2.12). By standard results there is a sequence  $h_k \to 0$  and a function w with  $||w||_{L^2(\Omega')} \leq L$  such that  $[u(\gamma(h_k, x)) - u(x)]/h_k$  converges weakly in  $L^2(\Omega')$  to w(x). In particular, for every  $\varphi \in \operatorname{Lip}_0(\Omega')$ ,

$$\lim_{h_k \to 0} \int_{\Omega} \frac{u(\gamma(h_k, x)) - u(x)}{h_k} \varphi(x) \, dx = \int_{\Omega} w(x)\varphi(x) \, dx.$$
(2.13)

Let us show that  $Xu = V \cdot \nabla u$  exists in the weak sense in  $\Omega'$  and satisfies Xu = w in  $\Omega'$ , or equivalently (see (1.9)) that for all  $\varphi \in \operatorname{Lip}_0(\Omega')$ ,

$$\int_{\Omega} u(x) V(x) \cdot \nabla \varphi(x) \, dx + \int_{\Omega} u(x) \varphi(x) \operatorname{div} V(x) \, dx = -\int_{\Omega} w(x) \varphi(x) \, dx$$

Since  $V \in C^1(\Omega)$ , then  $\gamma(t, x)$  is continuously differentiable in x by [1]. By approximation, we may also assume that  $\varphi$  is continuously differentiable in  $\Omega'$ ; in fact, any  $\varphi \in \operatorname{Lip}_0(\Omega')$  can be approximated in ordinary  $W^{1,2}(\Omega')$  norm by a smooth approximation to the identity that is still supported in  $\Omega'$ .

For simplicity, we will write  $h = h_k$  and  $h \to 0$  instead of  $h_k \to 0$ , and we consider only those h which are small enough depending on  $\Omega'$  to make the argument below valid. By (2.13),

$$\frac{1}{h} \int_{\Omega} u(\gamma(h, x))\varphi(x) \, dx - \frac{1}{h} \int_{\Omega} u(x)\varphi(x) \, dx \to \int_{\Omega} w\varphi \, dx \quad \text{as } h \to 0.$$

In the first integral, make the change of variables  $z = \gamma(h, x)$  with h fixed and small, and recall from (2.11) that the inverse transformation is  $x = \gamma(-h, z)$ . Using (2.5) and (2.6), we obtain

$$\begin{split} &\frac{1}{h}\int_{\Omega}u(z)\varphi(\gamma(-h,z))\{1+\text{trace }D(-h,z)+O(h^2)\}\,dz-\frac{1}{h}\int_{\Omega}u(z)\varphi(z)\,dz\\ &\rightarrow\int_{\Omega}w\varphi\,dz\quad\text{as }h\rightarrow0. \end{split}$$

Note that after the change of variables, we are able to retain  $\Omega$  as the domain of integration in the first integral due to the support of  $\varphi(\gamma(-h, z))$ , namely, if  $\varphi(\gamma(-h, z)) \neq 0$ , then  $\gamma(-h, z) \in \Omega'$ , and consequently  $z \in \Omega$  if |h| is small enough depending on  $\Omega'$ . Therefore, by regrouping terms,

$$\int_{\Omega} u(z) \frac{\varphi(\gamma(-h,z)) - \varphi(z)}{h} dz + \int_{\Omega} u(z)\varphi(\gamma(-h,z)) \frac{\operatorname{trace} D(-h,z)}{h} dz \to \int_{\Omega} w\varphi dz; \quad (2.14)$$

here we have used the estimate

$$\begin{split} & \frac{1}{|h|} O(h^2) \left| \int_{\Omega} u(z) \varphi(\gamma(-h,z)) \, dz \right| \\ & \leq O(h) \|u\|_{L^2(\Omega)} \bigg( \int_{\Omega} |\varphi(\gamma(-h,z))|^2 dz \bigg)^{\frac{1}{2}} \\ & \leq O(h) \|u\|_{L^2(\Omega)} \bigg( \int_{\Omega} |\varphi(x)|^2 \big\{ 1 + O(h) \big\} \, dx \bigg)^{\frac{1}{2}} \to 0 \quad \text{ as } h \to 0. \end{split}$$

Let us show that as  $h \to 0$  the first integral on the left in (2.14) converges to  $-\int_{\Omega} u V \cdot \nabla \varphi \, dz$  and the second integral there converges to  $-\int_{\Omega} u \varphi \, \text{div} V \, dz$ . In the first integral, as in Lemma 2.1,

$$\frac{\varphi(\gamma(-h,z)) - \varphi(z)}{h} = \frac{1}{h} \int_0^{-h} (\nabla \varphi)(\gamma(t,z)) \cdot V(\gamma(t,z)) \, dt$$

which converges pointwise to  $-V(z) \cdot \nabla \varphi(z)$  as  $h \to 0$  since  $\nabla \varphi$  is continuous. Thus, since

$$\left|\frac{\varphi(\gamma(-h,z)) - \varphi(z)}{h}\right| \le C \frac{|\gamma(-h,z) - z|}{|h|} \le C,$$

the part of the claim about the first integral in (2.14) follows from Lebesgue's Dominated Convergence Theorem. Similarly, if  $V = (v_1, \ldots, v_n)$ , then

trace 
$$D(-h,z) = \int_0^{-h} \left[ \sum_{j=1}^n \left( \frac{\partial}{\partial z_j} \gamma(s,z) \right) \cdot (\nabla v_j)(\gamma(s,z)) \right] ds$$

and by letting  $h \to 0$ , we obtain

$$\lim_{h \to 0} \frac{1}{h} \operatorname{trace} D(-h, z) = \lim_{h \to 0} \frac{1}{h} \int_0^{-h} \left[ \sum_{j=1}^n \left( \frac{\partial}{\partial z_j} \gamma(s, z) \right) \cdot (\nabla v_j) (\gamma(s, z)) \right] ds = -\operatorname{div} V(z)$$

since when s = 0,  $\frac{\partial}{\partial z_j}\gamma(s,z)$  is the unit vector  $e_j = (0,\ldots,0,1,0,\ldots,0)$  with *j*-th entry 1,  $j = 1,\ldots,n$ . This proves the lemma except for the assertion in its last sentence, which now follows by letting  $\Omega' \nearrow \Omega$ .

We remark that hypothesis (2.12) can be slightly weakened without changing the conclusion of Lemma 2.2, namely, in (2.12), the supremum over all h with  $0 < |h| < \delta$  can be replaced by the supremum over any sequence of h values converging to 0, and then the sequence  $\{h_k\}$  in the conclusion is some subsequence of the one used to form the supremum. In passing, note that if a function  $u \in L^2(\Omega)$  satisfies the hypothesis of Lemma 2.2 for every row vector  $V_i$  of  $\sqrt{Q}$  and for the same subset  $\Omega'$ , then the part of the conclusion of the lemma concerning  $\Omega'$  holds for every  $X_i u = V_i \cdot \nabla u$  with a sequence  $\{h_k\}$  that is independent of *i*. Then  $u \in H^{1,2}_{\mathcal{X}}(\Omega')$ , where  $\mathcal{X} = \{X_i\}_{i=1}^n$ . Since  $\sqrt{Q} \in \operatorname{Lip}(\Omega)$ , it follows (see the end of the Introduction) that  $u \in W^{1,2}_Q(\Omega')$ . Furthermore,  $\|\sqrt{Q}\nabla u\|_{L^2(\Omega')} \leq \sum_{i=1}^n L_i$ , where

$$L_i = \left(\sup_{h:0<|h|<\delta} \int_{\Omega'} \left|\frac{u(\gamma(V_i,h,x)) - u(x)}{h}\right|^2 dx\right)^{\frac{1}{2}}.$$

If every  $L_i$  is bounded by a number L that is independent of  $\Omega'$ , then  $u \in W^{1,2}_Q(\Omega)$  and  $\|\sqrt{Q}\nabla u\|_{L^2(\Omega)} \leq nL$ .

## 3 Part of the Proof of Theorem 1.1

In this section we begin the proof of Theorem 1.1, completing its more technical parts in the next section. Let  $\sqrt{Q} \in C^{1,1}(\Omega)$ ,  $u, b \in W^{1,2}_Q(\Omega)$ , and u be a weak solution in  $\Omega$  of  $\operatorname{div}(Q(x)\nabla u) = b(x)$ , namely,

$$\int_{\Omega} (Q(x)\nabla u) \cdot \nabla \Phi(x) \, dx + \int_{\Omega} b(x)\Phi(x) \, dx = 0 \quad \text{for all } \Phi \in \operatorname{Lip}_0(\Omega).$$
(3.1)

Let  $\Omega'$  be an open set with compact closure in  $\Omega$ , and let  $\gamma(h, x) = \gamma(V, h, x)$  for any fixed row  $V = V_i$  of  $\sqrt{Q}$ . Our strategy is to use (3.1) and (1.7) to prove that there exists  $\delta > 0$  such that

$$\sup_{h:|h|<\delta} \int_{\Omega'} \left| \frac{(\sqrt{Q}\nabla u)(\gamma(h,x)) - (\sqrt{Q}\nabla u)(x)}{h} \right|^2 dx \le L < \infty$$

with L independent of  $\Omega'$ . It then follows from Lemma 2.2 that  $\sqrt{Q}\nabla u \in W_Q^{1,2}(\Omega)$ , or more precisely, each of its components  $V_i \cdot \nabla u \in W_Q^{1,2}(\Omega)$ , proving Theorem 1.1.

Fix a row  $V = (v_1, \ldots, v_n)$  of  $\sqrt{Q}$  and let  $\gamma(h, x) = \gamma(V, h, x)$ . If  $\psi$  is any function defined in  $\Omega$ , denote

$$(\Delta^h \psi)(x) = \frac{\psi(\gamma(h, x)) - \psi(x)}{h} \quad \text{if } x, \gamma(h, x) \in \Omega.$$

We wish to replace  $\Phi$  in (3.1) by  $\Delta^{-h}\varphi(x)$  for a function  $\varphi$  supported in  $\Omega$  to be chosen. As noted earlier, there may be no fixed h such that  $\Delta^{-h}\varphi(x)$  is defined for every  $x \in \Omega$  due to the requirement that  $\gamma(-h, x) \in \Omega$ . This technical difficulty can be overcome by a simple convention. First, extend V to  $\mathbb{R}^n$  as a Lipschitz function  $V^* = (v_1^*, \ldots, v_n^*)$  with the same Lipschitz constant and consider the integral curves  $\gamma^*(h, x) = \gamma(V^*, h, x), (h, x) \in \mathbb{R} \times \mathbb{R}^n$ . Next, assuming that  $\varphi$  has support in  $\Omega$ , extend  $\varphi$  to  $\mathbb{R}^n$  by setting it equal to 0 outside its support. Denoting the extension again by  $\varphi$ , we define

$$(\Delta^{h,*}\varphi)(x) = \frac{\varphi(\gamma^*(h,x)) - \varphi(x)}{h}, \quad (h,x) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n,$$

and replace  $\Phi$  in (3.1) by  $(\Delta^{-h,*}\varphi)(x)$ , noting that  $\varphi(\gamma^*(-h,x))$  has support in  $\Omega$  if h is small enough depending on the support of  $\varphi$ . Then (3.1) takes the form

$$-\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\nabla[\varphi(\gamma^{*}(-h,x))]\,dx + \frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx$$
$$=\frac{1}{h}\int_{\Omega}b(x)\,\varphi(\gamma^{*}(-h,x))\,dx - \frac{1}{h}\int_{\Omega}b(x)\,\varphi(x)\,dx.$$
(3.2)

The right side of (3.2) becomes, after changing variables in the first integral there,

$$\frac{1}{h} \int_{\Omega} b(\gamma^*(h, x))\varphi(x)\{1 + O(h)\} dx - \frac{1}{h} \int_{\Omega} b(x)\varphi(x) dx$$

$$= \int_{\Omega} \left(\Delta^{h,*}b\right)(x)\varphi(x) dx + \int_{\Omega} b(\gamma^*(h, x))\varphi(x)O(1) dx.$$
(3.3)

The operation of translation along an integral curve does not generally commute with the gradient. Let us compute  $\nabla[\varphi(\gamma^*(-h, x))]$ . Assuming as we may that  $\varphi \in C^1(\Omega)$ , we have

$$\nabla[\varphi(\gamma^*(-h,x))] = \left( (\nabla\varphi)(\gamma^*(-h,x)) \cdot \frac{\partial}{\partial x_1} \gamma^*(-h,x), \dots, (\nabla\varphi)(\gamma^*(-h,x)) \cdot \frac{\partial}{\partial x_n} \gamma^*(-h,x) \right),$$

and for  $j = 1, \ldots, n$ ,

$$\frac{\partial}{\partial x_j}\gamma^*(-h,x) = \left(\frac{\partial}{\partial x_j}\gamma_1^*(-h,x),\ldots,\frac{\partial}{\partial x_j}\gamma_n^*(-h,x)\right).$$

Since  $(d/dt)\gamma_i^*(t,x) = v_i^*(\gamma^*(t,x))$ , then as before,

$$\gamma_i^*(-h,x) = x_i + \int_0^{-h} v_i^*(\gamma^*(s,x)) \, ds \quad \text{and}$$
$$\frac{\partial}{\partial x_j} \gamma_i^*(-h,x) = \delta_{ij} + \int_0^{-h} (\nabla v_i^*)(\gamma^*(s,x)) \cdot \frac{\partial}{\partial x_j} \gamma^*(s,x) \, ds.$$

Collecting estimates, we obtain the formula

$$\nabla[\varphi(\gamma^*(-h,x))] = (\nabla\varphi)(\gamma^*(-h,x)) + \tilde{D}^*(-h,x)(\nabla\varphi)(\gamma^*(-h,x)),$$
(3.4)

where  $\tilde{D}(t, x)$  is the transpose of the matrix D(t, x) in (2.7) and  $\tilde{D}^*$  is its analogue with  $\gamma^*, V^*$  in place of  $\gamma, V$ .

Let us now further specify the support of  $\varphi$ . Choose open sets  $\Omega'', \Omega'''$  depending on  $\Omega'$  with  $\Omega' \subset \subset \Omega'' \subset \subset \Omega'' \subset \subset \Omega$  and let  $\varphi$  be supported in  $\Omega''$ . Substituting (3.4) in the first integral on the left side of (3.2) shows that the left side of (3.2) equals

$$\begin{aligned} &-\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot(\nabla\varphi)(\gamma^{*}(-h,x))\,dx\\ &-\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\tilde{D}^{*}(-h,x)(\nabla\varphi)(\gamma^{*}(-h,x))\,dx\\ &+\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx.\end{aligned}$$

In the first integral here, let  $x = \gamma^*(h, z)$  and use the fact that  $\gamma^*(-h, \gamma^*(h, z)) = z$  to rewrite the expression as

$$\begin{aligned} &-\frac{1}{h}\int_{\Omega}Q(\gamma^*(h,z))(\nabla u)(\gamma^*(h,z))\cdot\nabla\varphi(z)\left\{1+O(h)\right\}dz\\ &-\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\tilde{D}^*(-h,x)(\nabla\varphi)(\gamma^*(-h,x))\,dx\\ &+\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx.\end{aligned}$$

Note that the domain of integration in the first term remains  $\Omega$  if h is small due to the support of  $\varphi$ . The domains may also be replaced by  $\Omega'''$  if h is small enough depending on  $\Omega'', \Omega'''$ . Furthermore, if h is small enough (depending on  $\Omega'''$ ), then  $\gamma^*(s, x) = \gamma(s, x)$  for all  $x \in \Omega'''$  and all s with  $|s| \leq |h|$ . If h satisfies these restrictions, we can drop all asterisks above, obtaining the equivalent expression

$$\begin{aligned} &-\frac{1}{h}\int_{\Omega}\left(\sqrt{Q}\nabla u\right)(\gamma(h,z))\cdot\left(\sqrt{Q(\gamma(h,z))}-\sqrt{Q(z)}\right)\nabla\varphi(z)\left\{1+O(h)\right\}dz\\ &-\frac{1}{h}\int_{\Omega}\left(\sqrt{Q}\nabla u\right)(\gamma(h,z))\cdot\left(\sqrt{Q}\nabla\varphi\right)(z)\left\{1+O(h)\right\}dz\\ &-\frac{1}{h}\int_{\Omega}Q(x)\nabla u(x)\cdot\tilde{D}(-h,x)(\nabla\varphi)(\gamma(-h,x))\,dx\\ &+\frac{1}{h}\int_{\Omega}\left(\sqrt{Q}\nabla u\right)(z)\cdot\left(\sqrt{Q}\nabla\varphi\right)(z)\,dz.\end{aligned}$$

Regrouping in the second and fourth terms and recalling (3.3), we may rewrite (3.2) as

$$I =: \int_{\Omega} \Delta^{h} \left( \sqrt{Q} \nabla u \right)(z) \cdot \left( \sqrt{Q} \nabla \varphi \right)(z) dz - \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(\gamma(h, z)) \cdot \left( \Delta^{h} \sqrt{Q} \right)(z) \nabla \varphi(z) \{1 + O(h)\} dz - \frac{1}{h} \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(\gamma(h, z)) \cdot \left( \sqrt{Q} \nabla \varphi \right)(z) O(h) dz - \frac{1}{h} \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(x) \cdot \sqrt{Q(x)} \tilde{D}(-h, x) (\nabla \varphi)(\gamma(-h, x)) dx - \left[ \int_{\Omega} \left( \Delta^{h} b \right)(x) \varphi(x) dx + \int_{\Omega} b(\gamma(h, x)) \varphi(x) O(1) dx \right] = II + III + IV + V,$$
(3.5)

with constants independent of h and x in the O(h), O(1) factors if  $|h| < \delta(\Omega', V)$ .

Now let  $\{u_k\}_{k=1}^{\infty}$  be a sequence of smooth functions representing  $(u, \nabla u)$  in  $W_Q^{1,2}(\Omega)$ , and let  $\eta$  be a smooth cutoff function supported in  $\Omega''$  and equal to 1 on  $\Omega'$ . Choose  $\varphi(x) = \eta(x)^2(\Delta^h u_k)(x)$  for each k, with the understanding that h is small enough (depending on  $\Omega'$ ) that  $\gamma(x,h) \in \Omega$  if  $\eta(x) \neq 0$ . Then

$$\nabla \varphi(x) = \eta(x)^2 \nabla (\Delta^h u_k)(x) + (\Delta^h u_k)(x) 2\eta(x) \nabla \eta(x)$$

and

$$\nabla(\Delta^{h}u_{k})(x) = \frac{\nabla[u_{k}(\gamma(h,x))] - \nabla u_{k}(x)}{h}$$
$$= \frac{1}{h} \{ (\nabla u_{k})(\gamma(h,x)) + \tilde{D}(h,x)(\nabla u_{k})(\gamma(h,x)) - \nabla u_{k}(x) \}$$
$$= (\Delta^{h}\nabla u_{k})(x) + \frac{1}{h}\tilde{D}(h,x)(\nabla u_{k})(\gamma(h,x)).$$

Combining these formulas gives

$$\nabla\varphi(x) = \eta(x)^2 (\Delta^h \nabla u_k)(x) + \eta(x)^2 \frac{1}{h} \tilde{D}(h, x) (\nabla u_k) (\gamma(h, x)) + (\Delta^h u_k)(x) 2\eta(x) \nabla \eta(x).$$
(3.6)

Hence,

$$\begin{split} \left(\sqrt{Q}\nabla\varphi\right)(x) &= \eta(x)^2\sqrt{Q(x)}(\Delta^h\nabla u_k)(x) \\ &+ \eta(x)^2\sqrt{Q(x)}\frac{1}{h}\tilde{D}(h,x)(\nabla u_k)(\gamma(h,x)) + (\Delta^h u_k)(x)\,2\eta(x)\big(\sqrt{Q}\nabla\eta\big)(x). \end{split}$$

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We will use this to rewrite I. First note that

$$\sqrt{Q(x)}(\Delta^h \nabla u_k)(x) = \frac{1}{h} \sqrt{Q(x)}(\nabla u_k)(\gamma(h, x)) - \frac{1}{h} \left(\sqrt{Q} \nabla u_k\right)(x)$$
$$= \Delta^h \left(\sqrt{Q} \nabla u_k\right)(x) - \Delta^h \left(\sqrt{Q}\right)(x)(\nabla u_k)(\gamma(h, x))$$

by adding and subtracting  $\frac{1}{h} (\sqrt{Q} \nabla u_k) (\gamma(h, x))$ . Substituting this identity into the first term on the right in the previous formula for  $(\sqrt{Q} \nabla \varphi)(x)$  gives

$$\left(\sqrt{Q}\nabla\varphi\right)(x) = \eta(x)^2 \Delta^h \left(\sqrt{Q}\nabla u_k\right)(x) - \eta(x)^2 \Delta^h \left(\sqrt{Q}\right)(x)(\nabla u_k)(\gamma(h, x)) + \eta(x)^2 \sqrt{Q(x)} \frac{1}{h} \tilde{D}(h, x)(\nabla u_k)(\gamma(h, x)) + (\Delta^h u_k)(x) 2\eta(x) \left(\sqrt{Q}\nabla\eta\right)(x).$$
(3.7)

Therefore, in addition to (3.5) we have the formula

$$I = \int_{\Omega} \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \cdot \eta(x)^{2} \Delta^{h} \left( \sqrt{Q} \nabla u_{k} \right)(x) dx$$
  

$$- \int_{\Omega} \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \cdot \eta(x)^{2} \Delta^{h} \left( \sqrt{Q} \right)(x) (\nabla u_{k}) (\gamma(h, x)) dx$$
  

$$+ \int_{\Omega} \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \cdot \eta(x)^{2} \sqrt{Q(x)} \frac{1}{h} \tilde{D}(h, x) (\nabla u_{k}) (\gamma(h, x)) dx$$
  

$$+ \int_{\Omega} \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \cdot (\Delta^{h} u_{k})(x) 2\eta(x) \left( \sqrt{Q} \nabla \eta \right)(x) dx$$
  

$$= I_{1} + I_{2} + I_{3} + I_{4}.$$
(3.8)

Note that  $\lim_{k\to\infty} I_1$  equals

$$\int_{\Omega} \left| \Delta^h \left( \sqrt{Q} \nabla u \right)(x) \right|^2 \eta(x)^2 dx \ge \int_{\Omega'} \left| \Delta^h \left( \sqrt{Q} \nabla u \right)(x) \right|^2 dx \tag{3.9}$$

since  $\eta = 1$  on  $\Omega'$ . Therefore, by the strategy outlined near the beginning of the section, it will be enough to show that the  $\limsup_{k\to\infty}$  of the absolute value of each of I<sub>2</sub>, I<sub>3</sub>, I<sub>4</sub> and II, III, IV, V (with  $\varphi = \eta^2 \Delta^h u_k$  in II, III, IV, V) is dominated by terms that can be absorbed into the integral on the left (or right) side of (3.9) or are bounded in h. Assumption (1.7) in Theorem 1.1 helps to accomplish this. The hypothesis that  $\sqrt{Q} \in C^{1,1}(\Omega)$  is used heavily in estimating II.

The remaining computations are lengthy and postponed until the next section.

### 4 The Remainder of the Proof of Theorem 1.1

This section is devoted to completing the proof of Theorem 1.1. It remains to derive the estimates mentioned at the end of Section 3 for I<sub>2</sub>, I<sub>3</sub>, I<sub>4</sub> and II, III, IV, V. Recall that the function  $\varphi$  in II, III, IV, V is always  $\varphi = \eta^2 \Delta^h u_k$  where  $\{u_k\}$  is a sequence of Lipschitz functions representing u in  $W_Q^{1,2}(\Omega)$ . The number of estimates is large, but some of the computations are similar to one another. The assumption that  $\sqrt{Q} \in C^{1,1}(\Omega)$  is used to estimate II. We begin with a fact about Lipschitz continuity in t of integral curves  $\gamma(t, x)$ .

**Lemma 4.1** Let V(x) be a Lipschitz vector field in  $\Omega$  and  $\gamma(t,x) = \gamma(V,t,x)$ . If J is an interval in  $\mathbb{R}$  such that  $\gamma(t,x)$  is defined for all  $t \in J$  and  $|J| ||V||_{\text{Lip}(\Omega)} \leq \frac{1}{2}$ , then for all  $t_1, t_2 \in J$ ,

$$|\gamma(t_1, x) - \gamma(t_2, x)| \le \int_J |V(\gamma(t, x))| \, dt \le 2|J| \min_{t \in J} |V(\gamma(t, x))|.$$
(4.1)

*Proof* Let  $t_1, t_2 \in J$ . The first inequality in (4.1) is true because

$$|\gamma(t_1, x) - \gamma(t_2, x)| = \left| \int_{t_2}^{t_1} \gamma'(t, x) \, dt \right| \le \int_J |V(\gamma(t, x))| \, dt.$$

For any  $t_0 \in J$ ,

$$\begin{split} \int_{J} |V(\gamma(t,x))| \, dt &\leq \int_{J} |V(\gamma(t,x)) - V(\gamma(t_0,x))| \, dt + |J| \, |V(\gamma(t_0,x))| \\ &\leq \int_{J} \|V\|_{\operatorname{Lip}(\Omega)} |\gamma(t,x) - \gamma(t_0,x)| \, dt + |J| \, |V(\gamma(t_0,x))| \\ &\leq \int_{J} \|V\|_{\operatorname{Lip}(\Omega)} \int_{J} |V(\gamma(s,x))| \, ds \, dt + |J| \, |V(\gamma(t_0,x))|, \end{split}$$

where we have used the first inequality in (4.1). The last sum equals

$$|J| \|V\|_{\operatorname{Lip}(\Omega)} \int_{J} |V(\gamma(s, x))| \, ds + |J| \, |V(\gamma(t_0, x))|,$$

and by subtracting its first summand from  $\int_J |V(\gamma(t,x))| dt$  and using the assumption that  $|J| ||V||_{\text{Lip}(\Omega)} \leq 1/2$ , we obtain

$$\int_{J} |V(\gamma(t,x))| dt \le 2|J| |V(t_0,x)|$$

Since  $t_0$  is an arbitrary point of J, the second inequality in (4.1) follows, completing the proof of the lemma.

Next we list some corollaries of Lemma 4.1.

**Lemma 4.2** Let V,  $\gamma(t, x)$  and J be as in Lemma 4.1. If  $F : \Omega \to \mathbb{R}^k$  is a Lipschitz vector in  $\mathbb{R}^k$ , then for all  $t_1, t_2 \in J$ ,

$$\left| F(\gamma(t_1, x)) - F(\gamma(t_2, x)) \right| \le \|F\|_{\operatorname{Lip}(\Omega)} 2|J| \min_{t \in J} |V(\gamma(t, x))|,$$
(4.2)

$$\frac{1}{|J|} \int_{J} \left| F(\gamma(t,x)) \right| dt \le \min_{t \in J} \left| F(\gamma(t,x)) \right| + \|F\|_{\operatorname{Lip}(\Omega)} 2|J| \min_{t \in J} |V(\gamma(t,x))|.$$
(4.3)

*Proof* It suffices to prove (4.2), since (4.3) then follows by integration. Since

$$\left|F(\gamma(t_1, x)) - F(\gamma(t_2, x))\right| \le \|F\|_{\operatorname{Lip}(\Omega)} |\gamma(t_1, x) - \gamma(t_2, x)|,$$

(4.2) follows by applying (4.1).

We also record an obvious analogue of (4.2) for matrices: If M(x) is a Lipschitz matrix on  $\Omega$  mapping  $\mathbb{R}^n$  into  $\mathbb{R}^k$ , and if  $V, \gamma$  and J are as in Lemma 4.1, then for all  $t_1, t_2 \in J$  and all  $\xi \in \mathbb{R}^n$ ,

$$\left| M(\gamma(t_1, x))\xi - M(\gamma(t_2, x))\xi \right| \le \|M\|_{\operatorname{Lip}(\Omega)} 2|J| \min_{t \in J} |V(\gamma(t, x))| \, |\xi|.$$
(4.4)

Here, the notation  $||M||_{\text{Lip}(\Omega)}$  means for example the sum of the Lipschitz constants on  $\Omega$  of the row vectors of M.

A typical application of (4.4) that will help to estimate  $I_2$  is

$$\left| \left( \Delta^h \sqrt{Q} \right)(x) \nabla u_k(\gamma(h, x)) \right| \le C \left\| \sqrt{Q} \right\|_{\operatorname{Lip}(\Omega)} |V(\gamma(h, x))| |\nabla u_k(\gamma(h, x))|, \tag{4.5}$$

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with C independent of x and small h. In fact,

$$\left| \left( \Delta^{h} \sqrt{Q} \right)(x) \nabla u_{k}(\gamma(h, x)) \right| = \left| \frac{1}{h} \left\{ \sqrt{Q(\gamma(h, x))} - \sqrt{Q(x)} \right\} \nabla u_{k}(\gamma(h, x)) \right|$$
$$\leq \left\| \sqrt{Q} \right\|_{\operatorname{Lip}(\Omega)} C \frac{|h|}{|h|} |V(\gamma(h, x))| |\nabla u_{k}(\gamma(h, x))| \quad \text{by (4.4) with } \xi = \gamma(h, x)$$

Therefore (see (3.8)),

$$\begin{aligned} |\mathbf{I}_{2}| &\leq C \int_{\Omega} \left| \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \right| \eta(x)^{2} \left| V(\gamma(h,x)) \right| \left| \nabla u_{k}(\gamma(h,x)) \right| dx \\ &\leq C \epsilon \int_{\Omega} \left| \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \right|^{2} \eta(x)^{2} dx + C \frac{1}{\epsilon} \int_{\Omega} \left| V(\gamma(h,x)) \right|^{2} \left| \nabla u_{k}(\gamma(h,x)) \right|^{2} \eta(x)^{2} dx \end{aligned}$$

for any  $\epsilon > 0$ . By picking  $\epsilon$  small, the first term in the last expression can be absorbed into the integral on the right side of (3.9). The second term, after making the change of variables  $z = \gamma(h, x)$ , is bounded for small h by

$$C\int_{\Omega} |V(z)|^2 |\nabla u_k(z)|^2 dz, \qquad (4.6)$$

whose  $\limsup_{k\to\infty}$  is at most  $(\limsup_{k\to\infty} S_k)^2$  with  $S_k$  as in (1.6).

Next let us estimate  $|I_4|$ . By (3.8), with  $\eta$  fixed,

$$\begin{aligned} |\mathbf{I}_4| &\leq C \int_{\Omega} \left| \Delta^h \left( \sqrt{Q} \nabla u \right)(x) \right| |\eta(x)| \left| (\Delta^h u_k)(x) \right| dx \\ &\leq C \epsilon \int_{\Omega} \left| \Delta^h \left( \sqrt{Q} \nabla u \right)(x) \right|^2 \eta(x)^2 dx + C \frac{1}{\epsilon} \int_{\Omega''} |(\Delta^h u_k)(x)|^2 dx. \end{aligned}$$

The first integral immediately above can be absorbed into the integral on the left in (3.9). The second one is bounded by a multiple of  $\|\sqrt{Q}\nabla u_k\|_{L^2(\Omega)}^2$  by Lemma 2.1, and so its  $\limsup_{k\to\infty} S_k$  is at most  $C(\limsup_{k\to\infty} S_k)^2$ .

We now turn to estimating  $|I_3|$ . Note that for any  $\epsilon > 0$ ,

$$|\mathbf{I}_{3}| \leq \epsilon \int_{\Omega} \left| \Delta^{h} \left( \sqrt{Q} \nabla u \right)(x) \right|^{2} \eta(x)^{2} dx + C \frac{1}{\epsilon} \int_{\Omega''} \left| \sqrt{Q(x)} \frac{1}{h} \tilde{D}(h,x) \left( \nabla u_{k} \right)(\gamma(h,x)) \right|^{2} dx.$$
(4.7)

If  $\epsilon$  is small, the first integral on the right side can again be absorbed. In order to find a suitable upper bound for the second integral on the right, we apply (2.4) to obtain

$$\left|\frac{\partial}{\partial x_i}\gamma(t,x)\right| \le 2 \quad \text{if } x \in \Omega'' \text{ and } |t| < c_{V,\Omega''},$$

and therefore (see (2.7))

$$\left\|\frac{1}{h}\tilde{D}(h,x)\right\|_{\text{op}} \le C \quad \text{for } x \in \Omega'' \text{ and } |h| < c_{V,\Omega''}.$$
(4.8)

In our estimates, constants that depend on  $\Omega''$  are determined ultimately by  $\Omega'$ . Write

$$\sqrt{Q(x)} \frac{1}{h} \tilde{D}(h, x) (\nabla u_k) (\gamma(h, x))$$

$$= \left\{ \sqrt{Q(x)} - \sqrt{Q(\gamma(h, x))} \right\} \frac{1}{h} \tilde{D}(h, x) (\nabla u_k) (\gamma(h, x))$$

$$+ \sqrt{Q(\gamma(h, x))} \frac{1}{h} \tilde{D}(h, x) (\nabla u_k) (\gamma(h, x)) = T_1 + T_2.$$
(4.9)

Consider  $T_2$  first. By (4.8),

$$|T_2| \le C \left\| \sqrt{Q(\gamma(h,x))} \right\|_{\text{op}} \left| \left( \nabla u_k \right) (\gamma(h,x)) \right|$$

The contribution of  $T_2$  to the second integral in (4.7), namely its contribution to the integral

$$\int_{\Omega''} \left| \sqrt{Q(x)} \frac{1}{h} \tilde{D}(h, x) \left( \nabla u_k \right) (\gamma(h, x)) \right|^2 dx,$$
(4.10)

is then at most

$$C\int_{\Omega''} \left\|\sqrt{Q(\gamma(h,x))}\right\|_{\rm op}^2 \left|\left(\nabla u_k\right)(\gamma(h,x))\right|^2 dx.$$

The change of variables  $z = \gamma(h, x)$  shows that this is bounded (for small h) by

$$C \int_{\Omega} \left\| \sqrt{Q(x)} \right\|_{\text{op}}^{2} \left| \nabla u_{k}(x) \right|^{2} dx.$$

$$(4.11)$$

Since  $\|\sqrt{Q(x)}\|_{\text{op}} \leq C \sum_{\ell} |w_{\ell}(x)|$ , where  $\{w_{\ell}\}_{\ell}$  are the entries of  $\sqrt{Q}$ , the  $\limsup_{k \to \infty}$  of (4.11) is at most  $C(\limsup_{k \to \infty} S_k)^2$ .

In order to estimate the part of (4.10) corresponding to  $T_1$ , namely  $\int_{\Omega''} |T_1|^2 dx$ , note by (4.4) that

$$\begin{aligned} |T_1| &\leq C \|\sqrt{Q}\|_{\operatorname{Lip}(\Omega)} |h| |V(\gamma(h, x))| \left\| \frac{1}{h} \tilde{D}(h, x) \right\|_{\operatorname{op}} \left| (\nabla u_k) (\gamma(h, x)) \right| \\ &\leq C |h| |V(\gamma(h, x))| \left| (\nabla u_k) (\gamma(h, x)) \right|. \end{aligned}$$

Therefore, by changing variables as usual, we obtain

$$\int_{\Omega''} |T_1|^2 dx \le Ch^2 \int_{\Omega} |V(x)|^2 |\nabla u_k(x)|^2 dx,$$
(4.12)

another estimate whose  $\limsup_{k\to\infty}$  is bounded as above for small h.

It remains to estimate II, III, IV, V (see (3.5) in case  $\varphi(x) = \eta(x)^2 (\Delta^h u_k)(x)$ . Except for II, whose estimation will involve a novelty related to second differences, most of the arguments needed will be similar to ones already used. We begin with III. By (3.5),

$$|\mathrm{III}| \le C \int_{\Omega} \left| \left( \sqrt{Q} \nabla u \right) (\gamma(h, z)) \cdot \left( \sqrt{Q} \nabla \varphi \right) (z) \right| dz$$

Using formula (3.7) to rewrite  $(\sqrt{Q}\nabla\varphi)(z)$ , it follows that if  $\epsilon > 0$ , then

$$\begin{aligned} |\mathrm{III}| &\leq C\frac{1}{\epsilon} \int_{\Omega} \left| \left( \sqrt{Q} \nabla u \right) (\gamma(h,z)) \right|^{2} \eta(z)^{2} dz + C\epsilon \int_{\Omega} \left| \Delta^{h} \left( \sqrt{Q} \nabla u_{k} \right) (z) \right|^{2} \eta(z)^{2} dz \\ &+ C \bigg( \int_{\Omega} \left| \left( \sqrt{Q} \nabla u \right) (\gamma(h,z)) \right|^{2} \eta(z)^{2} dz \bigg)^{\frac{1}{2}} \bigg( \int_{\Omega''} \left| \Delta^{h} \left( \sqrt{Q} \right) (z) (\nabla u_{k}) (\gamma(h,z)) \right|^{2} dz \bigg)^{\frac{1}{2}} \\ &+ C \bigg( \int_{\Omega} \left| \left( \sqrt{Q} \nabla u \right) (\gamma(h,z)) \right|^{2} \eta(z)^{2} dz \bigg)^{\frac{1}{2}} \bigg( \int_{\Omega''} \left| \sqrt{Q(z)} \frac{\tilde{D}(h,z)}{h} (\nabla u_{k}) (\gamma(h,z)) \right|^{2} dz \bigg)^{\frac{1}{2}} \\ &+ C \bigg( \int_{\Omega} \left| \left( \sqrt{Q} \nabla u \right) (\gamma(h,z)) \right|^{2} \eta(z)^{2} dz \bigg)^{\frac{1}{2}} \bigg( \int_{\Omega''} \left| (\Delta^{h} u_{k}) (z) 2 \eta(z) (\sqrt{Q} \nabla \eta) (z) \right|^{2} dz \bigg)^{\frac{1}{2}}. \end{aligned}$$

By the usual change of variables, the first term on the right side above is at most a constant times  $\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}^2 \leq (\limsup_{k\to\infty} S_k)^2$ , and the limit as  $k\to\infty$  of the second term can be absorbed as usual provided  $\epsilon$  is chosen to be small. For each of the last three terms, we have

already noted that the first factor is bounded by  $C \limsup_{k\to\infty} S_k$ , and the corresponding second factors have been estimated in the arguments for I<sub>2</sub>, I<sub>3</sub> and I<sub>4</sub> respectively. This completes our estimation of III.

In order to estimate term II in (3.5), apply  $\Delta^h \sqrt{Q}$  to formula (3.6) to obtain

$$\begin{split} \left(\Delta^{h}\sqrt{Q}\right)(z)\nabla\varphi(z) &= \eta(z)^{2}\left(\Delta^{h}\sqrt{Q}\right)(z)(\Delta^{h}\nabla u_{k})(z) \\ &+ \eta(z)^{2}\left(\Delta^{h}\sqrt{Q}\right)(z)\frac{1}{h}\tilde{D}(h,z)\left(\nabla u_{k}\right)(\gamma(h,z)) \\ &+ \left(\Delta^{h}u_{k}\right)(z)2\eta(z)\left(\Delta^{h}\sqrt{Q}\right)(z)\nabla\eta(z), \end{split}$$

and in the first term on the right side of this equation, rewrite

$$\begin{split} \left(\Delta^h \sqrt{Q}\right)(z) (\Delta^h \nabla u_k)(z) &= \frac{1}{h} \frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h} (\nabla u_k) (\gamma(h,z)) \\ &- \frac{1}{h} \frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h} \nabla u_k(z). \end{split}$$

Combining equalities gives a representation of II as four integrals:

$$\begin{split} -\mathrm{II} &= \int_{\Omega} \left( \sqrt{Q} \nabla u \right) (\gamma(h, z)) \cdot \eta(z)^2 \\ &\times \frac{1}{h} \bigg( \frac{\sqrt{Q}(\gamma(h, z))}{h} - \sqrt{Q(z)}}{h} \bigg) \big( \nabla u_k \big) (\gamma(h, z)) \left\{ 1 + O(h) \right\} dz \\ &- \int_{\Omega} \left( \sqrt{Q} \nabla u \big) (\gamma(h, z)) \cdot \eta(z)^2 \frac{1}{h} \bigg( \frac{\sqrt{Q}(\gamma(h, z))}{h} - \sqrt{Q(z)}}{h} \bigg) \nabla u_k(z) \left\{ 1 + O(h) \right\} dz \\ &+ \int_{\Omega} \left( \sqrt{Q} \nabla u \big) (\gamma(h, z)) \cdot \eta(z)^2 \big( \Delta^h \sqrt{Q} \big) (z) \frac{1}{h} \tilde{D}(h, z) \big( \nabla u_k \big) (\gamma(h, z)) \left\{ 1 + O(h) \right\} dz \\ &+ \int_{\Omega} \big( \sqrt{Q} \nabla u \big) (\gamma(h, z)) \cdot \big( \Delta^h u_k \big) (z) 2\eta(z) \big( \Delta^h \sqrt{Q} \big) (z) \nabla \eta(z) \left\{ 1 + O(h) \right\} dz. \end{split}$$

Of these four terms, we will consider only the first two since the sizes of the last two can be estimated easily by using Hölder's inequality and earlier estimates. The first two are very similar in form. The difficulty in estimating them is due to the presence of a combined power  $h^2$  in the denominators of their integrands. Each of the first two integrals generates two similar integrals, one arising from the "1" in the factor 1 + O(h) and the other arising from the "O(h)". We may consider only the two integrals associated with 1's since in the two associated with O(h)'s, there is partial cancellation of the power  $h^2$  in the denominator, again allowing us to apply Hölder's inequality and earlier estimates. The two integrals with 1's contribute

$$\int_{\Omega} \left(\sqrt{Q}\nabla u\right) (\gamma(h,z)) \cdot \eta(z)^2 \frac{1}{h} \left(\frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h}\right) \left(\nabla u_k\right) (\gamma(h,z)) dz \tag{4.13}$$

$$-\int_{\Omega} \left(\sqrt{Q}\nabla u\right)(\gamma(h,z)) \cdot \eta(z)^2 \frac{1}{h} \left(\frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h}\right) \nabla u_k(z) \, dz. \tag{4.14}$$

These will be combined rather than considered separately. Leaving the second one as is, we change variables  $z = \gamma(-h, x)$ , or equivalently  $x = \gamma(h, z)$ , in (4.13) and rewrite it as

$$\int_{\Omega} \left(\sqrt{Q}\nabla u\right)(z) \cdot \eta(\gamma(-h,z))^2 \frac{1}{h} \left(\frac{\sqrt{Q(z)} - \sqrt{Q(\gamma(-h,z))}}{h}\right) \nabla u_k(z) \left\{1 + O(h)\right\} dz$$

Here, we may as before ignore the O(h) part of the factor 1 + O(h). Similarly, we may replace  $\eta(\gamma(-h,z))^2$  in the integrand by  $\eta(z)^2$  since  $\eta(\gamma(-h,z))^2 - \eta(z)^2 = O(h)$ . By combining the result with (4.14), our task becomes to estimate the difference

$$\int_{\Omega} \left(\sqrt{Q}\nabla u\right)(z) \cdot \eta(z)^2 \frac{1}{h} \left(\frac{\sqrt{Q(z)} - \sqrt{Q(\gamma(-h,z))}}{h}\right) \nabla u_k(z) \, dz \\ - \int_{\Omega} \left(\sqrt{Q}\nabla u\right)(\gamma(h,z)) \cdot \eta(z)^2 \frac{1}{h} \left(\frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h}\right) \nabla u_k(z) \, dz.$$

To this difference, we subtract and add the integral

$$\int_{\Omega} \left( \sqrt{Q} \nabla u \right)(z) \cdot \eta(z)^2 \frac{1}{h} \left( \frac{\sqrt{Q(\gamma(h,z))} - \sqrt{Q(z)}}{h} \right) \nabla u_k(z) \, dz$$

and then regroup terms to obtain

$$\int_{\Omega} \left(\sqrt{Q}\nabla u\right)(z) \cdot \eta(z)^2 \frac{2\sqrt{Q(z)} - \sqrt{Q(\gamma(h,z))} - \sqrt{Q(\gamma(-h,z))}}{h^2} \nabla u_k(z) dz - \int_{\Omega} \Delta^h \left(\sqrt{Q}\nabla u\right)(z) \cdot \eta(z)^2 \left(\Delta^h \sqrt{Q}\right)(z) \nabla u_k(z) dz.$$

$$(4.15)$$

Of these two integrals, it is enough to estimate the first one since the other one can be estimated by using an argument of  $\epsilon$ ,  $\epsilon^{-1}$  type as usual (cf. the argument for  $I_2$ ).

The entries of the matrix

are second differences

$$\frac{\sqrt{Q(\gamma(h,z))} + \sqrt{Q(\gamma(-h,z))} - 2\sqrt{Q(z)}}{h^2}$$

$$\frac{f(h) + f(-h) - 2f(0)}{h^2},$$
(4.16)

where  $f(h) = f(h, z) = w(\gamma(h, z))$  and w is a generic entry of  $\sqrt{Q}$ . We have

$$f(h) - f(0) = \int_0^h f'(t) dt,$$
  

$$f(-h) - f(0) = \int_0^{-h} f'(t) dt = -\int_0^h f'(t-h) dt,$$
  

$$f(h) + f(-h) - 2f(0) = \int_0^h \left[ f'(t) - f'(t-h) \right] dt.$$

Since

$$f'(t) = (\nabla w)(\gamma(t,z)) \cdot \gamma'(t,z) = (\nabla w)(\gamma(t,z)) \cdot V(\gamma(t,z)),$$

then

$$\begin{aligned} f'(t) - f'(t-h) &= (\nabla w)(\gamma(t,z)) \cdot V(\gamma(t,z)) - (\nabla w)(\gamma(t-h,z)) \cdot V(\gamma(t-h,z)) \\ &= (\nabla w)(\gamma(t,z)) \cdot \left\{ V(\gamma(t,z)) - V(\gamma(t-h,z)) \right\} \\ &+ \left\{ (\nabla w)(\gamma(t,z)) - (\nabla w)(\gamma(t-h,z)) \right\} \cdot V(\gamma(t-h,z)). \end{aligned}$$

Hence, by Lemma 4.2, since  $\nabla w$  is Lipschitz continuous and  $|t| \leq |h|$  and h is small,

$$|f'(t) - f'(t - h)| \le |(\nabla w)(\gamma(t, z))| ||V||_{\operatorname{Lip}(\Omega)} 2|h| |V(z)| + ||\nabla w||_{\operatorname{Lip}(\Omega)} 2|h| |V(z)| |V(\gamma(t - h, z))| = O(h) |V(z)|.$$

Therefore,

$$\frac{f(h) + f(-h) - 2f(0)}{h^2} \bigg| = \frac{1}{h^2} \bigg| \int_0^h \left[ f'(t) - f'(t-h) \right] dt \bigg|$$
$$\leq \frac{1}{h^2} |h| O(h) |V(z)| \leq C |V(z)|.$$

Applying Schwarz's inequality to the first integral in (4.15) shows that its absolute value is at most a constant times

$$\begin{aligned} \|\sqrt{Q}\nabla u\|_{L^{2}(\Omega)} \left(\int_{\Omega''} \left|\frac{2\sqrt{Q(z)} - \sqrt{Q(\gamma(h,z))} - \sqrt{Q(\gamma(-h,z))}}{h^{2}} \nabla u_{k}(z)\right|^{2} dz\right)^{\frac{1}{2}} \\ &\leq C \|\sqrt{Q}\nabla u\|_{L^{2}(\Omega)} \left(\int_{\Omega} |V(z)|^{2} |\nabla u_{k}(z)|^{2} dz\right)^{\frac{1}{2}}, \end{aligned}$$

$$\tag{4.17}$$

with C depending on the sum of the Lipschitz constants of  $w_{\ell}$  and  $\nabla w_{\ell}$  (where  $\{w_{\ell}\}$  are the entries of  $\sqrt{Q}$ ) and on  $\|V\|_{L^{\infty}(\Omega)}$ . Estimate (4.17) is independent of h and its  $\limsup_{k\to\infty}$  is bounded by the right side of (1.8).

Next, let us derive an estimate for IV (see (3.5)) in case  $\varphi(x) = \eta(x)^2 (\Delta^h u_k)(x)$ . We will be brief since similar techniques have appeared earlier. As usual (see (3.6)),

$$\begin{aligned} \nabla \varphi(z) &= \eta(z)^2 \frac{1}{h} \big( \nabla u_k \big) (\gamma(h, z)) - \eta(z)^2 \frac{1}{h} \nabla u_k(z) \\ &+ \eta(z)^2 \frac{1}{h} \tilde{D}(h, z) \big( \nabla u_k \big) (\gamma(h, z)) + \big( \Delta^h u_k \big) (z) 2 \eta(z) \nabla \eta(z). \end{aligned}$$

Setting  $z = \gamma(-h, x)$  in this formula and noting that  $\gamma(h, \gamma(-h, x)) = x$  and that

$$(\Delta^h u_k)(\gamma(-h,x)) = (\Delta^{-h} u_k)(x),$$

we obtain

$$\begin{split} \big(\nabla\varphi\big)(\gamma(-h,x)\big) &= \eta(\gamma(-h,x))^2 \frac{1}{h} \nabla u_k(x) - \eta(\gamma(-h,x))^2 \frac{1}{h} \big(\nabla u_k\big)(\gamma(-h,x)) \\ &+ \eta(\gamma(-h,x))^2 \frac{1}{h} \tilde{D}(h,\gamma(-h,x)) \nabla u_k(x) \\ &+ \big(\Delta^{-h} u_k\big)(x) 2\eta(\gamma(-h,x)) (\nabla\eta)(\gamma(-h,x)). \end{split}$$

Denoting  $M(x) = M_h(x) = \sqrt{Q(x)}\tilde{D}(-h,x)/h$ , we may then rewrite IV in terms of four integrals:

$$\begin{split} -\mathrm{IV} &= \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(x) \cdot M(x) \eta(\gamma(-h,x))^2 \frac{1}{h} \nabla u_k(x) \, dx \\ &- \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(x) \cdot M(x) \eta(\gamma(-h,x))^2 \frac{1}{h} \big( \nabla u_k \big)(\gamma(-h,x) \big) \, dx \\ &+ \int_{\Omega} \left( \sqrt{Q} \nabla u \big)(x) \cdot M(x) \eta(\gamma(-h,x))^2 \frac{1}{h} \tilde{D}(h,\gamma(-h,x)) \nabla u_k(x) \, dx \\ &+ \int_{\Omega} \left( \sqrt{Q} \nabla u \big)(x) \cdot \big( \Delta^{-h} u_k \big)(x) 2 \eta(\gamma(-h,x)) M(x) (\nabla \eta)(\gamma(-h,x)) \, dx \\ &= \mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3 + \mathrm{IV}_4. \end{split}$$

In order to estimate IV<sub>4</sub>, it will be enough to use the simple fact that the entries of M(x) are bounded in x and h. Then Schwarz's inequality implies that if h is small, IV<sub>4</sub> is at most a

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constant times

$$\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|\Delta^{-h}u_k\|_{L^2(\Omega''')} \le C\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\|\sqrt{Q}\nabla u_k\|_{L^2(\Omega)},$$

where we have also applied Lemma 2.1. The  $\limsup_{k\to\infty}$  is at most  $C(\limsup_{k\to\infty} S_k)^2$ .

Term  $IV_3$  can also be estimated easily by using Schwarz's inequality and the sort of computations used for  $I_3$  (see for example (4.10)).

The expression  $IV_1 + IV_2$  will be estimated as a single entity. After setting  $z = \gamma(-h, x)$  in  $IV_2$ , we obtain

$$IV_1 + IV_2 = \frac{1}{h} \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(x) \cdot \eta(\gamma(-h, x))^2 M(x) \nabla u_k(x) \, dx - \frac{1}{h} \int_{\Omega} \left( \sqrt{Q} \nabla u \right)(\gamma(h, x)) \cdot \eta(x)^2 M(\gamma(h, x)) \nabla u_k(x) \left\{ 1 + O(h) \right\} dx.$$

The idea of the remaining computation is to combine the vector  $h^{-1}(\sqrt{Q}\nabla u)(x)$  from IV<sub>1</sub> with the vector  $-h^{-1}(\sqrt{Q}\nabla u)(\gamma(h,x))$  from IV<sub>2</sub> to obtain  $-\Delta^h(\sqrt{Q}\nabla u)(x)$  and then to apply the usual absorption technique involving  $\epsilon, \epsilon^{-1}$ . For this to work well, the matrix  $\eta(\gamma(-h,x))^2 M(x)$ in IV<sub>1</sub> and the matrix  $\eta(x)^2 M(\gamma(h,x))$  in IV<sub>2</sub> must be replaced by a coincident matrix, namely  $\eta(x)^2 M(x)$ , and the integral that arises from the O(h) part of 1+O(h) in IV<sub>2</sub> must be estimated separately. In fact, the integral that arises from the O(h) part of IV<sub>2</sub> is easy by itself because the O(h) factor cancels  $h^{-1}$  and then Schwarz's inequality and arguments related to (4.10) yield familiar size estimates; recall that  $M(x) = \sqrt{Q(x)}\tilde{D}(-h,x)/h$  while the similar matrix  $\sqrt{Q(x)}\tilde{D}(h,x)/h$  was considered earlier.

In order to replace both matrices  $\eta(\gamma(-h, x))^2 M(x)$  and  $\eta(x)^2 M(\gamma(h, x))$  by the same matrix  $\eta(x)^2 M(x)$ , we subtract and add  $\eta(x)^2 M(x)$  to the matrices in both integrals. In the integrand of the first integral, this produces the extra matrix

$$\left(\eta(\gamma(-h,x))^2 - \eta(x)^2\right)M(x) = O(h)M(x),$$

which due to cancellation of powers of h yields an integral that can again be estimated by using arguments related to (4.10). On the other hand, in the integrand of the second integral, we obtain the extra matrix  $\eta(x)^2 \{M(\gamma(h, x)) - M(x)\}$ , whose operator norm is bounded by a multiple of |h| |V(x)| since the entries of M are Lipschitz continuous. Due to cancellation of powers of h, the resulting integral is easy to estimate by Schwarz's inequality and has the familiar bound

$$C\|\sqrt{Q}\nabla u\|_{L^2(\Omega)}\||V|\nabla u_k\|_{L^2(\Omega)}.$$

This completes our estimation of all four parts of IV.

To estimate the size of term V, recall from (3.5) that

$$-\mathbf{V} = \int_{\Omega} \left( \Delta^h b \right)(x) \,\varphi(x) \, dx + \int_{\Omega} b(\gamma(h, x)) \varphi(x) \, O(1) \, dx.$$

Since  $\varphi = \eta(x)^2 (\Delta^h u_k)(x)$  with h small, Schwarz's inequality yields

$$\begin{aligned} |\mathbf{V}| &\leq C \|\Delta^{h} b\|_{L^{2}(\Omega)} \|\Delta^{h} u_{k}\|_{L^{2}(\Omega)} + C \|b\|_{L^{2}(\Omega)} \|\Delta^{h} u_{k}\|_{L^{2}(\Omega)} \\ &\leq C \|b\|_{W_{\Omega}^{1,2}(\Omega)} \|\sqrt{Q} \nabla u_{k}\|_{L^{2}(\Omega)} \quad \text{(by Lemma 2.1).} \end{aligned}$$

The last expression has  $\limsup_{k\to\infty}$  bounded by the right side of (1.8). This completes the estimates needed to prove Theorem 1.1, and the theorem now follows by collecting them.

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