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Some Developments in Nielsen Fixed Point Theory

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Abstract We give a brief survey of some developments in Nielsen fixed point theory. After a look at early history and a digress to various generalizations, we confine ourselves to several topics on fixed points of self-maps on manifolds and polyhedra. Special attention is paid to connections with geometric group theory and dynamics, as well as some formal approaches.

Keywords Fixed points, Nielsen theory

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1 Introduction

Nielsen fixed point theory is a research area in algebraic and geometric topology. The maps are allowed to deform continuously, in contrast to the metric fixed point theory originated from Banach's contraction principle. A central theme of Nielsen theory is the effect of the fundamental group on the behavior of fixed points of maps.

Nielsen fixed point theory is named after its founder Jakob Nielsen. In 1921, he determined the minimal number of fixed points in any isotopy class of self-homeomorphisms of the torus. This was immediately extended to self-maps of torus by Brouwer, their papers appeared side by side in *Math. Annalen.* Shortly after Lefschetz established his celebrated fixed point theorem which is homological in nature, Nielsen in 1927 published his seminal study [64] on self-homeomorphisms of hyperbolic surfaces, with special emphasis on the fixed point problem. He classifies fixed points according to their behavior on the universal covering space, or equivalently, in connection with the fundamental group. A fixed point class with non-zero index (= the sum of local winding numbers) is called essential because it can never disappear during a homotopy. So the number of essential fixed point classes (the Nielsen number) is a lower bound to the number of fixed points for all self-maps in that homotopy class.

Reidemeister and his student Wecken brought Nielsen's ideas into the mainstream of algebraic topology. Freed from the cumbersome language of non-Euclidean plane geometry used by Nielsen, the notions of fixed point class and Nielsen number became available for self-maps of

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polyhedra. Reidemeister 1936 [69] introduced a homotopy invariant (the Reidemeister trace) which combines algebraic coordinates and geometric indices of all fixed point classes, and bears a formal resemblance to the Lefschetz trace formula. It is theoretically computable once a cellular self-map is given. Wecken 1942 [76] proved his famous minimality theorem: the Nielsen number is the minimal number of fixed points for all self-maps in a given homotopy class, provided the underlying space is a manifold of dimension at least three. However, explicit computation of Nielsen number lagged behind for as long as four decades, except for Hopf 1927 on higher dimensional tori and projective spaces and Franz 1943 on three-dimensional lens spaces.

Chinese mathematicians made their contributions. Tsai-han Kiang, after his PhD degree under Marston Morse, worked as a research assistant of Lefschetz for one year before his return to China. In an effort to make Nielsen theory more accessible to topologists, his 1931 paper [56] replaced Nielsen's use of the circle at infinity of the non-Euclidean plane by infinite words in fundamental groups. In the early 1960's, Kiang organized a seminar on fixed point theory, participants including Jiang, Shi and You et al. Jiang introduced a class of spaces (called Jiang spaces) on which every loop is the track of a cyclic homotopy of the identity map. On such spaces the Nielsen number can be read off from homological information. Shi extended Wecken's minimality theorem to a much wider class of polyhedra. These developments were reviewed in a survey by Fadell [24], then presented in a book by Brown [13] and a more elementary book [57] by Kiang. These are signs of a renewal of interest in Nielsen fixed point theory.

Also in the 1970's, Nielsen's original study was reborn into the Nielsen–Thurston theory of surface diffeomorphisms, which has a profound impact in mathematics. This brings low dimensional problems back to the research front of fixed point theory.

The present article is a brief survey of developments thereafter. After a description of some theoretical generalizations, we confine ourselves to several aspects on fixed points of self-maps on manifolds and polyhedra. The exposition is sketchy. We used to mention initial and/or recent works on a topic, in the hope that the interested reader can trace the history from the references listed therein.

The Handbook [15] contains many more topics and literature than we are able to cover. For unexplained notions below, the reader may consult its Chapter III.16.

2 Theoretical Ramifications

The standard setting of fixed point theory considers maps $f : X \to X$ from a space to itself, and solutions $x \in X$ of the equation $x = f(x)$. Similar equations are everywhere in mathematics. Wherever a Lefschetz type index theory is available and an effect of the fundamental group is uncovered, a Nielsen type study would emerge.

2.1 Coincidence Theory

A coincidence of two maps $f, g : X \to Y$ is a point $x \in X$ such that $f(x) = g(x)$. Fixed point theory generalizes naturally to this setting where the domain and target spaces can be different.

Nielsen coincidence theory was initiated by Brooks and Brown [12]. See Gonçalves [15, Chapter I.1] for a more recent account. An extremely special case with g a constant map, called Nielsen root theory, already adds very interesting information to Hopf's degree theory on maps between manifolds of the same dimension.

In the smooth category, coincidences become intersections (between graphs of the maps embedded in the product manifold), a familiar topic in differential topology. For manifolds X and Y of the same dimension, the graphs of f and g have complementary dimensions and the coincidence index equals their algebraic intersection number. When X is of higher dimension than Y, bordism replaces the role of the index, see Koschorke's work [60].

Much of Nielsen coincidence theory takes inspiration from Nielsen fixed point theory. Although less broad, the fixed point setting has more structure: composition (self-maps form a monoid and have iterates) and commutation (compositions $X \xrightarrow{\phi} Y \xrightarrow{\psi} X$ and $Y \xrightarrow{\psi} X \xrightarrow{\phi} Y$ have identical fixed point behavior). It has closer connection to dynamics and algebra.

2.2 Relative Nielsen Theory

A pair of spaces (X, A) refers to a space and a subspace. Maps between pairs respect the designated subspaces. Schirmer [70] started to consider fixed points of self-maps on a pair of spaces. The definition of relative Nielsen number depends on the preferred location (e.g. on the whole X or only on $X \setminus A$) where fixed points are counted and minimized. See Zhao's account [15, Chapter III.18].

This basic relative setting can be nested in complicated ways, such as stratifications appearing in many branches of mathematics.

2.3 Parametrized and Fibrewise Fixed Point Theory

The behavior of fixed points during a homotopy/isotopy is of interest in Nielsen fixed point theory. A homotopy is a one-parameter family of self-maps. Two homotopic self-maps with equal number of fixed points may fail to be connected by a homotopy with the same number of fixed points.

Geoghegan and Nicas [30] proposed a parametrized Lefschetz–Nielsen fixed point theory. The r-parameter case leads to trace invariants in the r-th Hochschild homology groups, with the 0-parameter case reducing to the classical Reidemeister trace. The 1-parameter theory was developed in detail, and further extended [31] to a theory on closed orbits of flows in relation to the K_1 groups in algebraic K-theory.

The geometric side of the theory is less clear. When a homotopy on X is seen as a level-preserving self-map of the cylinder $X \times I$, the theory gives a lower bound to the number of path components of the fixed point set. But when is this lower bound sharp?

The parametrized theory fits into the framework of fibrewise topology. A fibrewise space X over a base space B means a continuous family of spaces X_b (fibres) with parameter $b \in B$, while a fibrewise self-map $f : X \to X$ sends each fibre into itself. Dold [21] has defined the fibrewise fixed point index. The impact of the fundamental group of B was considered [32] in the coincidence context.

2.4 Equivariant Fixed Point Theory

Equivariant topology studies spaces with a group action and maps respecting that action. Wong [78] gave a definition of equivariant Nielsen number by working through the isotropy stratification of the space, along with a Wecken type minimality theorem. See the very informative survey by Ferrario [15, Chapter II.8].

2.5 Other Directions

Traditionally, topological fixed point theory works on finite dimensional manifolds and complexes. Generalizations to infinite dimensions, multi-valued maps, etc., are motivated by potential applications to nonlinear analysis. Wherever Lefschetz theory can be generalized (under certain compactness assumptions), Nielsen theory can also, without extra obstacle.

Cotton-Clay [18] recently studied fixed points of area-preserving diffeomorphisms on surfaces via symplectic Floer homology. The relevant chain complex splits naturally into direct summands, one for each fixed point class. This puts Nielsen fixed point theory into a new perspective.

3 Computation and Estimation of Nielsen Number

The Nielsen number is geometrically defined. Its computation turns out to be difficult in general. Many simply stated questions remain unanswered. For example, Geoghegan's conjecture (1979): If $f : X \to X$ is a homotopy idempotent (in the sense that f is homotopic to f^2), then $N(f) \leq 1$. However, substantial progress has been made for several important classes of self-maps.

3.1 Homogeneous Spaces

For homogeneous spaces of the form G/H, where G is a compact Lie group and H is a Lie subgroup, Nielsen number can be computed with the help of the finite covering $G/H_e \to G/H$, where H_e is the connected component of H at the identity element e. Note that G/H_e is a Jiang space, and finite coverings are manageable (see a discussion by Jezierski [41]). An important tool is the product formula of Nielsen numbers for a fibred self-map (= self-map of a fibred space which sends each fibre into a possibly different fibre) initiated by Brown [14] and refined by You [80].

Anosov [2] and Fadell and Husseini [26] proved that for any self-map of a nilmanifold, the Nielsen number equals the absolute value of the Lefschetz number (this is called the Anosov relation). Moreover, the Lefschetz number can be read off from the induced endomorphism of Lie algebra. The Anosov relation, along with deviations from it, has since been studied on solvmanifolds, infra-nilmanifolds and infra-solvmanifolds, through the work of Heath, Keppelmann, McCord, Kim, Lee, Lee, Dekimpe and others. An interesting result is the averaging formula for Nielsen numbers [59].

From another view point, Wong [77] proved that when the homogeneous space G/H is orientable, all fixed point classes of a self-map are of the same sign. Hence the Nielsen number vanishes when the Lefschetz number vanishes, otherwise equals the Reidemeister number (which is algebraically defined, see the next paragraph). Clearly the converse of Lefschetz fixed point theorem holds on these spaces.

3.2 Reidemeister Trace and Twisted Conjugacy in Groups

The Reidemeister trace of a self-map $f : X \to X$ is a formal linear combination (with integer coefficients) of (twisted) conjugacy classes. Let $\phi : \pi_1(X) \to \pi_1(X)$ be the endomorphism induced by f. Two elements $\beta, \beta' \in \pi_1(X)$ are ϕ-conjugate if there exists $\alpha \in \pi_1(X)$ such that $\beta' = \phi(\alpha)\beta\alpha^{-1}$. Another way is to work in $\pi_1(T_f)$, the fundamental group of the mapping torus of f, obtained from $\pi_1(X)$ by adding a new generator z and new relations $z^{-1}\alpha z = \phi(\alpha)$ for all

$\alpha \in \pi_1(X)$. Then $\beta, \beta' \in \pi_1(X)$ are ϕ-conjugate if and only if $\beta z, \beta' z \in \pi_1(T_f)$ are conjugate in the ordinary sense. (When ϕ is an automorphism, $\pi_1(T_f)$ is the HNN extension of $\pi_1(X)$ relative to ϕ.)

The Reidemeister trace of a cellular map is not difficult to calculate. In particular, when X is a graph or a surface, Fadell and Husseini [25] wrote it down in terms of Fox calculus.

However, the passage from Reidemeister trace to Nielsen number depends on the ability to distinguish ϕ-conjugacy classes. Only when the Reidemeister trace is in its simplest form where terms with identical ϕ-conjugacy classes are already combined or cancelled, the number of terms is the Nielsen number and the coefficients are the indices of essential fixed point classes. So, if the conjugacy problem in $\pi_1(T_f)$ is decidable, the Nielsen number of f is algorithmically computable. Otherwise, linear representations of $\pi_1(T_f)$ can be used to obtain (often partial) information about conjugacy classes, for the purpose of estimating (from below) the Nielsen number.

3.3 Aspherical Polyhedra

For X a compact aspherical polyhedron ($\pi_i(X) = 0$ unless $i = 1$), the induced endomorphism $\phi : \pi_1(X) \to \pi_1(X)$ determines the homotopy type of the self-map f. Hence all homotopy invariants of f, such as Nielsen number, indices of essential fixed point classes, are determined by ϕ. Their computation deserves more attention in geometric group theory.

4 Algorithms for Nielsen Numbers on Surfaces and Graphs

4.1 Surface Self-homeomorphisms

For surface self-homeomorphisms, Jiang and Guo [47] proposed a standard form which supplements Thurston canonical form with twist information along reducing curves. Nielsen number can be read off easily from this standard form.

Finding the Thurston canonical form for a given self-homeomorphism has been a hot topic in various contexts. The main breakthrough is Bestvina and Handel's train-track approach [4] that yields an algorithmic proof of Thurston's classification theorem. (See Brinkmann [11] for an implementation.) It is not clear whether this approach can be extended to determine the standard form (and hence the Nielsen number).

Using his special technique for minimizing the number of fixed points by isotopy, Kelly [55] has given an algorithm to compute Nielsen number for surface self-homeomorphisms.

From another point of view, Préaux [67, 68] has shown that all 3-manifold groups have a solvable conjugacy problem. Hence Reidemeister trace always leads to Nielsen number for surface self-homeomorphisms.

If a self-map of a surface is not homotopic to a self-homeomorphism, it suffices to look at its restriction on a spine, so the problem reduces to one on graphs.

4.2 Self-maps of Graphs

Bestvina and Handel [3] developed a theory of free group automorphisms analogous to the Thurston theory of surface automorphisms. Starting with a self-homotopy equivalence of a finite graph, they use an algorithm of geometric moves, such as foldings and tightenings, to simplify it until a most efficient form is reached. The latter, called a relative train track map or RTT, is a filtration of subgraphs preserved by the self-map, such that in each layer the map either permutes the edges or stretches all edges by a factor $\lambda > 1$. With the help of a supplementary algorithm by Turner that calculates indivisible Nielsen paths (or INP, which joins fixed points of the same class), a complete picture of fixed point classes will arise, whereby the Nielsen number is determined.

From another view point, the conjugacy problem is solvable in free-by-cyclic groups (= HNN extension of free groups relative to automorphisms). See [6] or [10], both use RTT in an essential way. Hence Reidemeister trace works.

A general endomorphism of a free group (or equivalently, a general self-map of a graph) can be reduced by mutation (i.e., by a sequence of homotopies and commutations) to an injective endomorphism. For the latter the RTT algorithm still works (see [20]), yet no INP algorithm is known.

In a very different approach, starting with an endomorphism of free group, Wagner [73] introduced a purely combinatorial condition (called remnant condition) that restricts the amount of cancelation under the endomorphism, and gave an algorithm for Nielsen number under this condition. Focusing on the rank-two free group, Yi and Kim managed to handle various non-remnant cases and gave an algorithm [58] for any self-map of the figure-eight graph.

The algorithmic computation of Nielsen number on graphs is yet to be completed. The problem for arbitrary self-maps with rank > 2 remains open.

Besides the Nielsen number, the indices of essential fixed point classes and the ranks of fixed-element-subgroups in $\pi_1(X)$ are also interesting homotopy invariants of self-maps. As an application of Bestvina and Handel's theory, a delicate bound to these invariants was found for self-maps of compact surfaces and graphs [50].

5 Minimum Number of Fixed Points Under Homotopy

Wecken's minimality theorem has been extended [42] to self-maps of compact connected polyhedra that have no local cut points and are not surfaces with negative Euler characteristic. In the case of surfaces with negative Euler characteristic, Jiang [44] found that the Nielsen number is not always attainable as the minimum number of fixed points for self-maps. Then Zhang [81] and Kelly [52] showed that the discrepancy can even be arbitrary large on a given surface. Since then, the determination of the minimal fixed point number on surfaces stands out as a challenging problem.

The results of Jiang and Zhang are based on braid equations. The work [45] pushed this idea further and converted the problem into an algebraic minimizing problem of a width function in a certain free subgroup of the pure 2-braid group. Although the width is known to be algorithmically computable (see [35, Chapter 1, §7]), the minimization is still unsolved. Note that the braid idea has been successful in finding minimum root numbers of surface maps, see [5].

Kelly took an entirely different approach and gave a geometric algorithm for the minimal fixed point number [53], for self-maps of bounded surfaces.

For polyhedra with local cut points, the minimal fixed point number is calculated successfully only for the homotopy class of the identity map, by Shi [71], see [57, Chapter 4] for an exposition in English.

6 Minimum Number of Fixed Points Under Isotopy

Originally Nielsen worked on closed orientable hyperbolic surfaces and studied self-maps that induce automorphisms on fundamental group. He conjectured that any fixed point class might be made either to vanish or to reduce to a single point by homotopy, see [64, p. 293 of English translation]. This is finally given a detailed proof in [47]. Note that when the surface is bounded, we must either use a relative Nielsen number in stead, or allow self-embeddings in order to get away from boundary.

Jiang et al. [49] showed that the same conclusion is also true for orientation-preserving homeomorphisms on closed orientable 3-dimensional manifolds which are either Haken or geometric. Now that Thurston's Geometrization Conjecture has been proved, the above class includes all irreducible, orientable, closed 3-manifolds. The case of reducible 3-manifolds is open.

Kelly [54] has given a proof of attaining the Nielsen number through isotopy for homeomorphisms on manifolds of dimension at least 5, using the fact that any null-homotopic simple closed curve in such a manifold bounds an embedded disc. The failure of the latter fact is a typical difficulty in dimension four.

7 Periodic Points

Periodic points of a self-map $f : X \to X$ are fixed points of iterates of f. A point $x \in X$ is a periodic point with period $n \geq 1$ if $x = f^n(x)$ but $x \neq f^k(x)$ for all $0 < k < n$.

7.1 Nielsen Type Numbers and Their Realization

The simplest question to ask is about the number of periodic points. Halpern [37, 38], just before he left mathematics, worked towards finding a Nielsen type number for a self-map which is (the best) lower bound for the number of n-periodic points in its homotopy class. Later, Jiang [43, Section III.4] formulated the definition of Nielsen type numbers $NF_n(f)$ and $NP_n(f)$ which are lower bounds for the number of fixed points of f^n, and the number of n-periodic points of f, respectively. Both are defined in terms of the fixed point class data (their coordinates and indices) of iterates of f, hence are homotopy invariants. Twenty years later, a Wecken type minimality theorem was proved by Jezierski [40]: On manifolds of dimension at least 3, for any given integer $n > 0$, $NF_n(f)$ is realizable as the number of fixed points of the n-th iterate, of a self-map g homotopic to f. Likewise is $NP_n(f)$ realizable.

Is there a single self-map g (homotopic to f) realizing $NF_n(f)$ and $NP_n(f)$ for all n? This is too good to be true in general, but nice exceptions do exist, such as orientation-preserving homeomorphisms on surfaces (Boyland [9, Theorem 2.4]). Obviously, a necessary condition for this simultaneous realization is the equality $NF_n(f) = \sum_{k|n} NP_k(f)$ for all $n \geq 1$. Self-maps of tori/nilmanifolds/solvmanifolds satisfy this condition, hence are good candidates to look at. See Keppelmann's discussion [15, Chapter I.3, §9].

7.2 Periodic Points of Smooth Self-maps

Differentiable dynamics is much richer than topological dynamics. In the study of periodic points the difference was first discovered by Shub and Sullivan [72]. For continuous self-maps, Dold [22] noticed that the sequence of indices $\{\text{ind}(f^k; x_0)\}_{k=1}^{\infty}$ at an isolated fixed point x_0 satisfies certain congruences. In contrast, for smooth self-maps this sequence is severely restricted (see [17]). Graff and Jezierski ([34]) took all these restrictions into account to define a new Nielsen type number $NJD_n(f)$ for any smooth self-map f on a closed smooth manifold, which is a homotopy invariant lower bound to the number of fixed points of f^n. This invariant turns out to be much larger than the invariant $NF_n(f)$ from the topological category even if the underlying manifold is simply-connected [33]. They also proved a Wecken type minimality theorem in smooth category.

7.3 Minimal Set of Periods

The set of periods of a self-map $f : X \to X$ is the set consisting of the periods of all periodic points of f. The minimal (under homotopy) set of periods of f is the intersection of the sets of periods, for all self-maps in the homotopy class of f. The interest in this set stems from the period doubling bifurcation and 'period three implies chaos' phenomena in one-dimensional dynamics.

By a careful analysis of the Nielsen numbers of iterates, the minimal set of periods are determined for self-maps on 2- and 3-dimensional tori, also obtained is a qualitative trichotomy for higher dimensional tori (see [1] and [48]). A discussion for nilmanifolds and solvmanifolds is in Marzantowicz's account [15, Chapter I.4].

7.4 Free Degree

Nielsen [65] considered the question of how long the iterates of a self-homeomorphism can remain fixed point free. The free degree of a (oriented) manifold M is the maximal integer m (or infinity) such that M has a (orientation-preserving) self-homeomorphism with no periodic point of period $< m$.

Nielsen showed the free degree of an oriented closed surface is $2g - 2$ when genus $g \geq 3$, and is 2 or 3 (it is 2 [19]) when $g = 2$. Wang [74] determined the free degree for closed surfaces without orientation restrictions.

For bounded surfaces, it is now known [79] that the free degree has a uniform (with respect to the number of boundary components) upper bound depending only on the genus $g \geq 2$. More detailed computations have been done in the thesis of Chas [16].

7.5 Zeta-functions

A self-map $f : X \to X$ generates a semi-flow on the mapping torus T_f (called the suspension flow of f). A periodic point of f corresponds to a periodic orbit of this flow, the period of the former equals the length of the latter, and the coordinate (in $\pi_1(X)$) of the former corresponds to the free homotopy class (in $\pi_1(T_f)$) of the latter. Fuller [29] was the first to bring Nielsen classes into the study of periodic orbits in dynamics.

Zeta function is a favorite way in dynamics to organize sequential data. Combining ideas of Milnor [63] and Fuller and using an abelianized version of Nielsen fixed point theory, Fried [28] introduced the Lefschetz zeta function that keeps track of homology information of all closed orbits, and related it to the Reidemeister torsion. Twisted Lefschetz zeta function can also be obtained through a linear representation of $\pi_1(T_f)$. It is a rational function carrying asymptotic information about the growth rate of Nielsen numbers and the topological entropy [46].

For a discussion on various zeta functions related to Nielsen fixed point theory, see Fel'shtyn's monograph [27].

8 Braid Forcing in Dynamics

An influential discovery in one-dimensional dynamics is the Sharkovskii order in the set of periods for self-maps on the interval. In the hope of finding a two-dimensional analogue, Matsuoka [62] and Boyland [7] started in early 1980's to consider an orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane with a given periodic orbit P of n points. For any isotopy $F := \{f_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{t\in I}$ from the identity map to f, the braid $\{(f_t(P), t)\} \subset \mathbb{R}^2 \times I$, modulo "full twists" and conjugacy in the braid group B_n, is independent of the isotopy F, and called the braid type of the orbit P. They noticed that for such homeomorphisms, the existence of (an orbit of) one braid type guarantees the co-existence of (orbits of) certain other braid types. This brings up a partial order among braid types, called 'braid forcing'. It has since been a hot and rich topic in dynamics, contributors include Boyland, Guaschi, Hall, Handel, Los, Matsuoka and many others. Theories and algorithms are developed about how the braid type determines geometric features of the planar homeomorphism, such as its set of periods, its Thurston canonical form, etc. See the survey paper by Boyland [8]. The relevance of Nielsen fixed point theory was first made clear in [39].

Jiang and Zheng [51] developed a theory of braid forcing in the framework of Nielsen fixed point theory. On a configuration space of the plane, the forced braids arise as coordinates of essential fixed point classes (as in the Reidemeister trace) of a self-homeomorphism (determined by the given braid). They even gave an algorithm which, for any braid type β and any given size m, produces a finite list of all braid types in B_m that are forced by β. Thus, the forcing order of braid types is algorithmically computable.

The importance of a mathematical notion or theory is often seen through its role in the interplay of different branches of mathematics. The braid forcing problem happens to be such a good test ground for Nielsen fixed point theory between topology and dynamics.

9 Formal Approaches

Now that categorification is making progress in many areas of mathematics, we take a look at formal approaches to Nielsen fixed point theory.

In view of a formal analogy, in algebraic geometry Grothendieck established a Lefschetz trace formula which played a basic role in Deligne's proof of the Weil conjecture. He also gave a Nielsen–Wecken formula [36] which resembles the Reidemeister trace, for an endomorphism on a variety over a finite field. Its significance remains to be explored.

Lück [61] constructed a universal (in a functorial sense) Lefschetz invariant for fixed points on CW complexes. Weber [75] developed it into an equivariant version of Reidemeister trace, and gave a converse of the equivariant Lefschetz fixed point theorem when this invariant vanishes.

Dold and Puppe [23] considered Lefschetz number as a trace associated to a dualisable object in a symmetric monoidal category. Hence, the Lefschetz fixed point theorem can be proved formally by using functoriality properties. Ponto [66] generalized this to a trace in bicategories with shadows, so that the Reidemeister trace and the converse of Lefschetz theorem are valid in more general (including fibrewise and parametrized) contexts.

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