Acta Mathematica Sinica, English Series
Jan., 2018, Vol. 34, No. 1, pp. 103–113
Published online: July 15, 2017
https://doi.org/10.1007/s10114-017-7048-8
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Semi-stable Extensions Over 1-dimensional Bases

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Abstract Given a family of Calabi–Yau varieties over the punctured disc or over the field of Laurent series, we show that, after a finite base change, the family can be extended across the origin while keeping the canonical class trivial. More generally, we prove similar extension results for families whose log-canonical class is semi-ample. We use these to show that the Berkovich and essential skeleta agree for smooth varieties over $\mathbb{C}((t))$ with semi-ample canonical class.

Keywords Semi-stable extension, Laurent series, essential skeleton

MR(2010) Subject Classification 14E30, 14J10

1 Introduction

Let C be a smooth curve, $C^\circ \subset C$ an open subset and $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$ a projective, locally stable family (see Definition 7) such that $K_{X^\circ} + \Delta^\circ$ is p°-nef. It is frequently useful to extend it to a projective, locally stable family $p : (X, \Delta) \to C$ such that $K_X + \Delta$ is p-nef.

This is not possible in general, but conjecturally such an extension exists after pulling back to a suitable finite (ramified) cover $B \to C$. Currently the latter is known if C is a curve defined over a field of characteristic 0, the fibers of p° over C° are divisorial log terminal and $K_{X^\circ} + \Delta^\circ$ is p°-semi-ample. In increasingly general forms this has been proved in [2, 3, 6, 9, 19].

In many applications, one would like to know such results when the smooth curve C is replaced by a 1-dimensional regular scheme or by a (non-compact) Riemann surface. The cases when $C = \operatorname{Spec} \mathbb{C}[[t]]$ or $C = \mathbb{D}$ (the complex unit disc) appear especially frequently.

The current proofs rely heavily on the Minimal Model Program, which is expected to hold in rather general settings. However, complete proofs are known only for schemes of finite type,

Received January 31, 2017, accepted May 24, 2017
Partial financial support was provided to JK by the NSF under grant number DMS-1362960, to JN by Starting Grant MOTZETA (Grant No. 306610) of the European Research Council and to CX by the Chinese National Science Fund for Distinguished Young Scholars (Grant No. 11425101)

although the relevant vanishing theorems have been established over 1-dimensional power series rings as well [21].

It is possible to approximate a general family by an algebraic family to arbitrary order and try to construct the extension by going systematically between the general family and its approximations. For Calabi–Yau families, this was worked out in [23, Sec. 4.2], but the general case seems to present technical difficulties.

Instead of approximation, we embed a 1-dimensional formal or analytic family into a higher dimensional algebraic family. Then we solve a special case of the algebraic extension problem over higher dimensional bases and induce the formal or analytic extension form it.

From now on all schemes are over a field of characteristic 0.

Notation 1 (Global extension problem) We work in one of the following set-ups.

(Algebraic case) C is a Noetherian, excellent, 1-dimensional, regular scheme, $C^\circ \subset C$ is a dense, open subscheme and $Z := C \setminus C^\circ$.

(Analytic case) C is a (not necessarily compact) Riemann surface and $C^\circ \subset C$ is an open subset with finite complement $Z := C \setminus C^\circ$.

In both cases we also have a projective morphism $p : (X, \Delta) \to C$ whose restriction $p^\circ : (X^\circ, \Delta^\circ) := (p^{-1}(C^0), \Delta|_{p^{-1}(C^0)}) \to C^\circ$ is locally stable with $\mathbb{Q}$-factorial, dlt fibers and $K_{X^\circ} + \Delta^\circ$ is p°-semi-ample.

The notions dlt, G-equivariantly dlt, qdlt and G-equivariantly qdlt are discussed in Definition 7; see [5, 12, 17] for more systematic treatments.

The precise extension results are the following.

Theorem 2 (Locally-stable version) *Using the assumptions of Notation* 1*, for every $c_i \in Z$, there are natural numbers $m(c_i)$ such that the following holds.*

Let $\tau : B \to C$ be a finite, surjective, Galois morphism such that $m(c_i)$ divides its ramification index over c_i for every $c_i \in Z$. Then there is a projective morphism $p_B : (X_B, \Delta_B) \to B$ with the following properties.

(1) *Over $B^\circ := \tau^{-1}(C^\circ)$, the morphism p_B is isomorphic to the pulled-back morphism $(X^\circ, \Delta^\circ) \times_C B^\circ \to B^\circ$,*

(2) $(X_B, \Delta_B + F_{Z,B})$ *is* dlt*, where $F_{Z,B} \subset X_B$ denotes the sum of the fibers of p_B over the points in $\tau^{-1}(Z)$, and*

(3) *$K_{X_B} + \Delta_B + F_{Z,B}$ is p_B-semi-ample.*

Furthermore, if G is a finite group acting on $p : (X, \Delta) \to C$ and on $\tau : B \to C$, then we can choose $p_B : (X_B, \Delta_B) \to B$ such that, in addition,

(4) *G_B acts on $p_B : (X_B, \Delta_B) \to B$, where G_B is the natural extension of G by* $\mathrm{Gal}(B/C)$ *and*

(5) $(X_B, \Delta_B + F_{Z,B})$ *is G_B-equivariantly* dlt.

Note that Theorem 2 (2) implies that p_B is locally stable along $F_{Z,B}$, in particular the fibers $F_{Z,B}$ are reduced. Together with (1) this implies that $p_B : (X_B, \Delta_B) \to B$ is locally stable. (Its fibers are semi-dlt in the terminology of [17, Sec. 5.4].) Furthermore, any fiber is a p_B-trivial divisor, hence (3) is equivalent to saying that $K_{X_B} + \Delta_B$ is p_B-semi-ample. For our applications it is important to know that, at least in the local case, the above model is obtained by an MMP from an snc model.

Complement 3 Using the above notation, assume in addition that C is local with closed point 0 and $C^\circ = C \setminus \{0\}$. Then one can choose $p_B : (X_B, \Delta_B) \to B$ with the following properties.

(1) There is a projective resolution $(X_B^1, \Delta_B^1) \to (X, \Delta) \times_C B$ such that the pair $(X_B^1, \Delta_B^1 + F_{Z,B}^1)$ is snc.

(2) There is a sequence $\phi_i : (X_B^i, \Delta_B^i) \dashrightarrow (X_B^{i+1}, \Delta_B^{i+1})$ of $K_{X_B^i} + \Delta_B^i$-negative contractions and flips starting with $i = 1$.

(3) The sequence terminates at (X_B, Δ_B) as in Theorem 2.

(4) The resolution and the sequence can be chosen to be $\mathrm{Gal}(B/C)$-equivariant.

Taking the quotient of $p_B : (X_B, \Delta_B) \to B$ by $\mathrm{Gal}(B/C)$ we obtain the following without base change.

Corollary 4 (Base change free version) *Using the assumptions of Notation* 1*, there is a projective morphism* $p_C : (X_C, \Delta_C) \to C$ *such that*

(1) *over* C° *it is isomorphic to* $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$,

(2) $(X_C, \Delta_C + \mathrm{red}(F_{Z,C}))$ *is* qdlt *where* $F_{Z,C} \subset X_C$ *denotes the sum of the fibers of* p_C *over the points in* Z *and*

(3) $K_{X_C} + \Delta_C + \mathrm{red}(F_{Z,C})$ *is* p_C*-semi-ample.*

Furthermore, if G *is a finite group acting on* $p : (X, \Delta) \to C$*, then we can choose* $p_C : (X_C, \Delta_C) \to C$ *such that, in addition,*

(4) G *acts on* $p_C : (X_C, \Delta_C) \to C$ *and*

(5) $(X_C, \Delta_C + \mathrm{red}(F_{Z,C}))$ *is* G*-equivariantly* qdlt.

Unlike in Theorem 2, the fibers $F_{Z,C}$ are not reduced and so $p_C : (X_C, \Delta_C) \to C$ is not locally stable. Most likely in Corollary 4 (2) one can replace qdlt with dlt. Note also that Corollary 4 (3) does not imply that $K_{X_C} + \Delta_C$ is numerically p_C-semiample since $\mathrm{red}(F_{Z,C})$ is not numerically p_C-trivial. The following more precise local variant follows from Complement 3.

Complement 5 Using the above notation, assume in addition that C is local with closed point 0 and $C^\circ = C \setminus \{0\}$. Then the following hold.

(1) There is a projective morphism $(X_C^1, \Delta_C^1) \to (X, \Delta)$ such that the pair $(X_C^1, \Delta_C^1 + \mathrm{red}\, F_{Z,C}^1)$ is toroidal and hence qdlt. (In fact, globally the quotient of an snc pair $(X_B^1, \Delta_B^1 + F_{Z,B}^1)$.)

(2) There is a sequence $\phi_i : (X_C^i, \Delta_C^i + \mathrm{red}\, F_{Z,C}^i) \dashrightarrow (X_C^{i+1}, \Delta_C^{i+1} + \mathrm{red}\, F_{Z,C}^{i+1})$ of $(K_{X_C^i} + \Delta_C^i + \mathrm{red}\, F_{Z,C}^i)$-negative contractions and flips starting with $i = 1$.

(3) The sequence terminates at (X_C, Δ_C) as in Corollary 4.

Comments 6 The above results are expected to hold if we only assume that the fibers over C° are proper, slc and with nef log canonical class. Our conditions are imposed by current limitations of the MMP.

If the fibers are not assumed $\mathbb{Q}$-factorial, then the arguments still work but the resulting $p_B : (X_B, \Delta_B) \to B$ is isomorphic (over B°) to a small $\mathbb{Q}$-factorialization of the pulled-back morphism $(X^\circ, \Delta^\circ) \times_C B^\circ \to B^\circ$.

By formal gluing theory (see, for instance, [20, 1.1]) it is enough to prove the Complements for the completion (or henselisation) of the local scheme $(0, C)$. We need this during the proof in Paragraph 18.

Note that in Complement 3 (2) and Complement 5 (2) we do not establish that the ϕ_i correspond to extremal rays (rather than faces) but most likely this can be arranged.

The proof of Theorem 2 shows that the expected extension theorems hold over higher dimensional bases, once the global existence theorem of the moduli space of higher dimensional varieties of general type is written up in [18]. In writing this note we focus on 1-dimensional bases since they are frequently used in the study of degenerations of Calabi–Yau varieties and there are complete references available for the background results needed.

One could argue that over a general base scheme local projectivity is more natural than global projectivity and our proof also works if p is only locally projective. In this case of course p_B is only locally projective.

It would be more natural to start with a morphism $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$ instead of $p : (X, \Delta) \to C$. This is allowed in the algebraic case where a projective family over C° extends to a projective family over C; one can easily induce such an extension from a Hilbert scheme. In the analytic case the difference is, however, substantial. We still get a morphism from C° to a Hilbert scheme but we need to know that it is meromorphic near $C \setminus C^\circ$. This is why we need to start with a projective (or at least proper and algebraic) family over C.

2 MMP for Locally Stable Families Over a Smooth Base

Definition 7 We follow the usual terminology of [12, 17] and assume that the characteristic is 0. Recall that "dlt" stands for divisorial log terminal and "slc" for semi-log-canonical.

Let B be a smooth scheme. A morphism $p : (X, \Delta) \to B$ is called *locally stable* if $K_{X/B} + \Delta$ is $\mathbb{Q}$-Cartier, p is flat and (X_b, Δ_b) is slc for every $b \in B$. The equivalent characterization Corollay 9 (2) will be especially convenient to work with.

(There is a general definition of locally stable morphisms over an arbitrary base B, but it is more complicated to state; see [16, 18].)

If G is a finite group acting on a dlt pair (X, Δ), then it called G-*equivariantly* dlt or G-dlt if for every irreducible divisor $D \subset \lfloor \Delta \rfloor$ and for every $g \in G$, either $g(D) = D$ or $g(D) \cap D = \emptyset$. In this case, the quotient $(X/G, \Delta/G)$ need not be dlt. These form the local models of *quotient*-dlt or qdlt pairs [5]. For practical purposes the following description, proved in [5], may be the most useful.

Definition-Claim 7.1 A log canonical pair (X, Δ) is qdlt iff there is an open subset $U \subset X$ such that

(a) $(U, \lfloor \Delta \rfloor|_U)$ is toroidal and

(b) every log canonical center of (X, Δ) meets U.

Finally, if $H \subset G$ is a normal subgroup, then $G_1 := G/H$ acts on a qdlt pair $(X/H, \Delta/H)$, these give local models of G_1-*equivariantly* qdlt or G_1-qdlt pairs.

Proposition 8 *Let* $(0 \in B)$ *be a smooth, local scheme of dimension* r, $D_1 + \cdots + D_r \subset B$ *an* snc *divisor such that* $D_1 \cap \cdots \cap D_r = \{0\}$ *and* $p : (X, \Delta) \to B$ *a morphism. The following are equivalent.*

(1) $K_{X/B} + \Delta$ *is* $\mathbb{Q}$*-Cartier,* p *is flat and* (X_0, Δ_0) *is* slc.

(2) *The pair* $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ *is* slc.

Proof We use induction on r. Both implications are trivial if $r = 0$.

Assume (2). Then $K_X + \Delta + p^*D_1 + \cdots + p^*D_r$ is $\mathbb{Q}$-Cartier hence so is $K_X + \Delta$. Set $Y := p^*D_r$. Adjunction (cf. [18, 4.9]) shows that $(Y, \Delta|_Y + p^*D_1|_Y + \cdots + p^*D_{r-1}|_Y)$ is slc, hence (X_0, Δ_0) is slc by induction.

Next we show that $p^*D_1, \ldots, p^*D_r$ form a regular sequence along X_0. First $Y = p^*D_r$ is a log canonical center hence reduced and S_2. By induction on r the restrictions $p^*D_1|_Y, \ldots, p^*D_{r-1}|_Y$ form a regular sequence. Thus $p^*D_1, \ldots, p^*D_r$ is a regular sequence along X_0, hence p is flat.

Conversely, assume that (1) holds. By induction $(Y, \Delta|_Y + p^*D_1|_Y + \cdots + p^*D_{r-1}|_Y)$ is slc hence inversion of adjunction (cf. [18, 4.9]) shows that $(X, \Delta + p^*D_1 + \cdots + p^*D_r)$ is slc. □

Corollary 9 *Let* B *be a smooth scheme and* $p : (X, \Delta) \to B$ *a morphism. The following are equivalent.*

(1) $p : (X, \Delta) \to B$ *is locally stable.*

(2) *For every* snc *divisor* $D \subset B$*, the pair* $(X, \Delta + p^*D)$ *is* slc.

Corollary 10 *Let* B *be a smooth scheme and* $p : (X, \Delta) \to B$ *a projective, locally stable morphism. Let* $\phi : (X, \Delta) \dashrightarrow (X^w, \Delta^w)$ *be a weak canonical model over* B (*cf.* [12, 3.50]).

Then $p^w : (X^w, \Delta^w) \to B$ *is also locally stable.*

Proof Let $D \subset B$ be an snc divisor. By Corollary 9, $(X, \Delta + p^*D)$ is slc and $p^w : (X^w, \Delta^w + (p^*D)^w) \to B$ is also a weak canonical model over B by [17, 1.28]. Thus $(X^w, \Delta^w + (p^*D)^w)$ is also slc. Next we claim that $(p^*D)^w = (p^w)^*D$. This is clear away from the exceptional set of ϕ^{-1} which has codimension ≥ 2 in X^w. Thus $(p^*D)^w$ and $(p^w)^*D$ are 2 divisors that agree outside a codimension ≥ 2 subset, hence they agree. Now we can use Corollary 9 again to conclude that $p^w : (X^w, \Delta^w) \to B$ is locally stable. □

11 (Relative MMP and base change) Let B be a smooth scheme, $p : (X, \Delta) \to B$ a projective, locally stable morphism and $\phi : (X, \Delta) \dashrightarrow (X', \Delta')$ be a $K_X + \Delta$-negative contraction or flip.

Let $p : C \to B$ be a morphism. By base change we get projective, locally stable morphisms $(X_C, \Delta_C) \to C$ and $(X'_C, \Delta'_C) \to C$. Is the induced morphism $\phi_C : (X_C, \Delta_C) \dashrightarrow (X'_C, \Delta'_C)$ a $K_{X_C} + \Delta_C$-negative contraction or flip?

The answer is clearly yes if ϕ is a morphism, but, if the relative Picard number changes, then ϕ_C could contract an extremal face. There are, however problems if ϕ is a flip. Let $Z' \subset X'$ denote the exceptional set if ϕ^{-1}. It has codimension ≥ 2, hence there is a closed, nowhere dense subset $W_1 \subset B$ such that $X'_b \cap Z'$ has codimension ≥ 2 for every $b \in B \setminus W_1$. Furthermore, we claim that $X'_b \cap Z'$ has codimension ≥ 1 for every $b \in B$. Indeed, choose an snc divisors $D_1, \ldots, D_r$ on B such that b is an irreducible component of $D_1 \cap \cdots \cap D_r$. Then every irreducible component of X'_b is an lc center of $(X', \Delta' + p'^*D_1 + \cdots + p'^*D_r)$ hence none of the irreducible components are contained in Z'.

Thus if $p(C) \subset W_1$, then Z'_C has codimension 1 in X'_C (hence ϕ_C is not a flip) but if $p^{-1}(W_1)$ is nowhere dense in C, then Z'_C has codimension ≥ 2 in X'_C and so ϕ_C is a flip.

Applying this to a whole sequence of contractions and flips gives the following.

Proposition 12 *Let B be a smooth scheme, $p : (X, \Delta) \to B$ a projective, locally stable morphism and $\phi : (X, \Delta) \dashrightarrow (X^w, \Delta^w)$ a composite of $K_X + \Delta$-negative contractions and flips. There is a closed, nowhere dense subset $W \subset B$ such that the following holds.*

Let C be an irreducible, smooth scheme and $p : C \to B$ a morphism whose image is not contained in W. Then, by base change we get

$$\phi_C : (X_C, \Delta_C) \dashrightarrow (X^w_C, \Delta^w_C)$$

which is also a composite of $K_{X_C} + \Delta_C$-negative contractions and flips.

Remark 13 Applying this to a weak log canonical model (X^w, Δ^w) over B and points $b \in B$, we see that the fiber $((X^w)_b, (\Delta^w)_b)$ of p^w is a weak canonical model of (X_b, Δ_b) for $b \in B \setminus W$ but not necessarily if $b \in W$. It is not easy to find conditions that imply that $W = \emptyset$, not even when $p^w : (X^w, \Delta^w) \to B$ is a minimal or canonical model. (See [8, Sec. 4] for some cases.) Thus, although the proof of Corollary 10 is short, the result seems quite surprising to us.

Combining Corollary 10, [9, 1.1] and [8, 2.9.2] we get the following.

Proposition 14 *Let B be a smooth, quasi-projective variety, $p : (X, \Delta) \to B$ a projective, locally stable morphism and H a p-ample divisor that does not contain any of the log canonical centers. Assume that general fibers are* dlt *and have a minimal model with semi-ample canonical class. Then, for $0 < \epsilon \ll 1$,*

(1) *the canonical model $(X^c, \Delta^c + \epsilon H^c)$ is independent of ϵ,*
(2) *$p^c : (X^c, \Delta^c + \epsilon H^c) \to B$ is locally stable,*
(3) *(X^c, Δ^c) is* dlt *and*
(4) *$K_{X^c} + \Delta^c$ is p^c-semiample.*

3 Locally Stable Extension of 1-dimensional Families

We start the proof of Theorem 2 with the local cases.

Notation 15 (Local extension problem) We work in one of the following set-ups.

(Algebraic case) Here (R, m) is a Noetherian, excellent, 1-dimensional regular local ring over a field of characteristic 0, $(c_0, C) := \mathrm{Spec}(R, m)$ and $C^\circ = C \setminus \{c_0\}$. We further have a projective morphism $p : (X, \Delta) \to C$. After restricting to C° we get $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$. We assume that the generic fiber of p° is dlt and $K_{X^\circ} + \Delta^\circ$ is $\mathbb{Q}$-Cartier and semi-ample.

(Analytic case) Here (c_0, C) is a small complex disc with origin c_0 and $C^\circ = C \setminus \{c_0\}$. We further have a projective morphism $p : (X, \Delta) \to C$. After restricting to C° we get $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$. We assume that the fibers of p° are dlt and $K_{X^\circ} + \Delta^\circ$ is $\mathbb{Q}$-Cartier and p°-semi-ample.

The construction of $p_B : (X_B, \Delta_B) \to B$ proceeds in several steps. We explain the algebraic case in detail and point out the instances where the analytic case is somewhat different.

16 (Semi-stable reduction) First we take a log resolution $\rho : (Y, \Delta_Y) \to (X, \Delta)$ such that $\Delta_Y + Y_0$ is an snc divisor where Y_0 denotes the central fiber and Δ_Y the sum of the components of $\rho_*^{-1}(\Delta_X) + \mathrm{Ex}(\rho)$ not contained in Y_0. For this the methods of Hironaka are sufficient; more precise references are [24] in the algebraic case and [14, 3.44] in the analytic case. These resolutions are also equivariant for any group action.

Furthermore, if H is an ample $\mathbb{Q}$-divisor on X, then we can assume that $H_Y := \rho^*H - E$ is ample on Y where E is a suitable, effective, ρ-exceptional $\mathbb{Q}$-divisor.

Let $n \geq 1$ be a common multiple of the multiplicities of the irreducible components of Y_0. For any $t \in m \setminus m^2$ set $C' := \operatorname{Spec}_C \mathcal{O}_C[x]/(x^n - t) \to C$. By base change we get $p' : (Y', \Delta'_Y) \to C'$ where Y' denotes the normalization of $Y \times_C C'$. Denote by Δ'_Y the pull back of Δ_Y. An easy local computation shows that Y'_0 is reduced and $(Y', \Delta'_Y + Y'_0)$ is toroidal, hence qdlt. Thus $p' : (Y', \Delta'_Y) \to C'$ is projective and locally stable with slc central fiber (Y'_0, Δ'_0) where $\Delta'_0 = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of Cartier divisors D_i.

To get the sharpest results, we use [11, Sec. IV.3] which says that, possibly after a further base change, we can find a log resolution $(Y'', \Delta''_Y) \to (Y', \Delta'_Y)$ such that $Y''_0 + \Delta''_Y$ is a reduced snc divisor where Δ''_Y denotes the birational transform of Δ'_Y. (Note that [11] discusses the algebraic case only, but the proof also works in the analytic setting.)

The pull-back of H_Y plus a relative ample divisor gives an ample $\mathbb{Q}$-divisor H''_Y on Y'' whose restriction to $Y'' \setminus Y''_0$ agrees with the pull-back of H_Y as $Y'' \to Y$ is finite outside the fiber over 0.

17 (Universal locally stable deformations) Let (Z, Δ) be a projective slc pair where $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of Cartier divisors D_i. For technical reasons we fix an ample divisor class on Z. We claim that its locally stable deformations have a universal deformation space.

First, Z has a universal deformation $Z^{\text{univ}} \to \operatorname{Def}(Z)$; see [10, Chap. III] for a discussion and references. One can choose $\operatorname{Def}(Z)$ to be a scheme of finite type and if a finite group acts on Z, then one can choose $Z^{\text{univ}} \to \operatorname{Def}(Z)$ to be G-equivariant.

By [1, 7, 15] it has a subspace $\operatorname{Def}^{\text{ls}}(Z) \subset \operatorname{Def}(Z)$ parametrizing those deformations where the relative canonical class is $\mathbb{Q}$-Cartier. Over this there is a universal family $\operatorname{CDiv} := \operatorname{CDiv}(Z^{\text{univ,ls}}/\operatorname{Def}^{\text{ls}}(Z))$ parametrizing Cartier divisors [13, I.1.13]. For each D_i we take a copy of CDiv and form the fiber product over $\operatorname{Def}^{\text{ls}}(Z)$. A neighborhood of the point corresponding to (Z, Δ) is a universal deformation space for $(Z, \sum a_i D_i)$. We denote this space by $\operatorname{Def}^{\text{ls}}(Z, \Delta)$. It comes with a universal family

$$p^{\text{univ}} : (X^{\text{univ}}, \Delta^{\text{univ}}) \to \operatorname{Def}^{\text{ls}}(Z, \Delta) \tag{17.1}$$

which is projective and locally stable.

(Note that in general $\operatorname{Def}^{\text{ls}}(Z, \Delta)$ does depend on the way we write Δ in the form $\Delta = \sum a_i D_i$. For instance if $\Delta = D$ is irreducible, then its deformations are again irreducible. If we write it as $\Delta = \frac{1}{2}D + \frac{1}{2}D$, then we allow deformations consisting of 2 divisors with coefficient $\frac{1}{2}$. This will not be a problem for us.)

18 (Algebraization of the original family) Using the central fiber (Y''_0, Δ''_0) of the family $p'' : (Y'', \Delta''_Y) \to C''$ obtained in Paragraph 16, we get $\operatorname{Def}^{\text{ls}}(Y''_0, \Delta''_0)$. After possibly replacing C'' by a suitable étale extension (without changing the residue field at the closed point), there is a natural moduli map $m''_C : C'' \to \operatorname{Def}^{\text{ls}}(Y''_0, \Delta''_0)$. Let $W'' \subset \operatorname{Def}^{\text{ls}}(Y''_0, \Delta''_0)$ denote the Zariski closure of the image. As we noted in Paragraph 6, it is enough to prove our claims for C''.

Next we choose a resolution of singularities $W \to W''$. By pulling back the universal family (17.1) to W, we obtain a projective, locally stable family

$$p_W : (Y_W, \Delta_W) \to W.$$

Our conditions were chosen to guarantee that the MMP (as in [3, 9]) runs and produces a minimal model

$$p_W^m : (Y_W^m, \Delta_W^m) \to W. \tag{18.1}$$

If the log canonical class is big on the generic fiber, we also get a canonical model

$$p_W^{\mathrm{c}} : (Y_W^{\mathrm{c}}, \Delta_W^{\mathrm{c}}) \to W. \tag{18.2}$$

19 (Local extensions) By construction, the family $p'' : (Y'', \Delta''_Y) \to C''$ is induced by base change by the quasi-finite moduli map $m''_C : C'' \to W$ whose image is Zariski dense. Thus, as we noted in Proposition 12, the MMP on $p_W : (Y_W, \Delta_W) \to W$ induces an MMP on $p'' : (Y'', \Delta''_Y) \to C''$ which ends with a minimal model.

For many applications this is quite satisfactory. We usually obtain the original family from some MMP, and there is no reason to favor one minimal model over another. However, when we want to glue the local extensions, it is important to know that we get the correct generic fiber. This can be achieved in at last 2 ways.

First, in the semi-stable reduction step (16) we can start with a thrifty log resolution (cf. [17, 2.87]). This ensures that when we construct the minimal model, it will be a small $\mathbb{Q}$-factorialization over the generic fiber. Thus if the generic fiber is $\mathbb{Q}$-factorial, we get back the generic fiber. This is why we assumed that the fibers are $\mathbb{Q}$-factorial.

Second, in Paragraph 16 we also obtained an ample $\mathbb{Q}$-divisor H''_Y. It is not effective, but we can replace it with a $\mathbb{Q}$-linearly equivalent effective ample divisor such that $(Y', \Delta' + H')$ is still dlt. We can use its new central fiber $(Y''_0, \Delta''_0 + H''_0)$ in Paragraph 18. By Proposition 14 the canonical model

$$p_W^{\mathrm{c}} : (Y_W^{\mathrm{c}}, \Delta_W^{\mathrm{c}} + \epsilon H^{\mathrm{c}}) \to W \tag{19.1}$$

exists, it is independent of $0 < \epsilon \ll 1$ and $K_{Y_W^{\mathrm{c}}} + \Delta_W^{\mathrm{c}}$ is p_W^{c}-semi-ample. If we use base change from (19.1), we get a family $p'' : (X'', \Delta'') \to C''$ which has the correct generic fiber. However, we can only guarantee that $(X'', \Delta'' + X''_0)$ is lc, not dlt.

20 (Global extensions) We now go back to the original set-up of Theorem 2.

We thus have a projective morphism $p : (X, \Delta) \to C$ whose restriction $p^\circ : (X^\circ, \Delta^\circ) \to C^\circ$ is locally stable with dlt fibers and $K_{X^\circ} + \Delta^\circ$ is p°-semi-ample. We may assume that C is integral. For each $c_i \in C \setminus C^\circ$ we have obtained natural numbers $m(c_i)$.

Let $\tau : B \to C$ be a finite, surjective, Galois morphism such that $m(c_i)$ divides its ramification index over c_i for every $c_i \in Z$. (It is easy to see that such morphisms exist. For instance, we start with such local maps $B_i \to (c_i, C)$, let $K \supset k(C)$ denote the Galois closure of the composite of the $k(B_i)$ and $\tau : B \to C$ the normalization of C in K.)

The previous local results give extensions of $p_B^\circ : (X^\circ, \Delta^\circ) \times_C B^\circ \to B^\circ$ locally over each c_i and these can be glued together to $p_B : (X_B, \Delta_B) \to B$.

As we discussed in Paragraph 19, we can keep track of a relatively ample divisor and ensure that p_B is projective.

21 (Equivariant versions) Assume next that a finite group acts on $p : (X, \Delta) \to C$. For each $s \in Z$ let $G_s \subset G$ denote the stabilizer of s. The resolution can be done G_s-equivariantly. We then use the normalization of C_s in the splitting field of $x^m - t$ to get $C''_s \to C_s$ with Galois group H_s. The canonical model is automatically $G_s \times H_s$-equivariant.

To achieve globalization, we let K be the Galois closure of the composite of all of the $k(C''_s)$ over $k(C/G)$. Set $H := \mathrm{Gal}(K/k(C/G))$ and $H_1 := \mathrm{Gal}(K/k(C))$. The actions on C give a natural isomorphism $G/G_1 \cong H/H_1$. Let $G_B \subset G \times H$ denote the preimage of the diagonal of $G/G_1 \times H/H_1$. Then G_B acts on $(X, \Delta) \times_C B \to B$ and our constructions are G_B-equivariant. Thus the resulting $p_B : (X_B, \Delta_B) \to B$ comes with a G_B-action.

22 (Extension without base change) We take the above $p_B : (X_B, \Delta_B) \to B$ with its G_B-action and set $(X_C, \Delta_C) := (X_B, \Delta_B)/H_1$. Note that the H_1-action may leave some irreducible components of the fibers fixed, so $X_B \to X_C$ can be ramified along some divisors in $F_{Z,B}$. Thus $K_{X_B} + \Delta_B$ need not be the pull-back of $K_{X_C} + \Delta_C$ but $K_{X_B} + \Delta_B + F_{Z,B}$ is the pull-back of $K_{X_C} + \Delta_C + F_{Z,C}$.

If the H_1-action has any fixed points, we can only guarantee that (X_C, Δ_C) has qdlt singularities. As explained in [5, Sec. 5], we should be able to pass to partial resolution with dlt singularities, but this again involves a version of the MMP that is not known in our case.

4 The Essential Skeleton of Degenerations

23 (Berkovich and essential skeleta) Let R be a complete, discrete valuation ring with residue field k and quotient field K. We assume that k has characteristic zero. For every K-scheme of finite type X° we denote by X^{an} its analytification in the sense of Berkovich. (See [4, 22] for introductions to Berkovich analytic spaces.) Let X be a regular, flat R-scheme of finite type such that the special fiber X_k has simple normal crossings and $X^\circ \cong X_K$. Then there is a canonical embedding of the dual intersection complex of X_k into X^{an}. The image of this embedding is called the *Berkovich skeleton* of X and denoted by $\mathrm{Sk}(X)$. If X is proper over R, then one can deduce from results of [4, 25] that $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}; see [23, Thm. 3.1.3] and the remarks after it.

Assume next that X° is a smooth, projective K-scheme and has a semi-ample canonical class K_{X°. The paper [21] defines the *essential skeleton* of $\mathrm{Sk}(X^\circ)$. It is a topological subspace of X^{an} with a piecewise integral affine structure obtained as a union of faces of the Berkovich skeleton $\mathrm{Sk}(X)$, but $\mathrm{Sk}(X^\circ)$ does not depend on the choice of X. It is proved in [23] that if X is a minimal, dlt model of X° over R, then $\mathrm{Sk}(X^\circ)$ is canonically homeomorphic to the dual intersection complex of X_k and it is a strong deformation retract of X^{an}. To guarantee the existence of such a minimal, dlt model X, [23] assumed that X° is defined over a k-curve. We will now remove this assumption using the results from the previous sections.

More generally, let X be a proper, flat R-scheme such that X_K is smooth and $(X, \mathrm{red}\, X_k)$ is qdlt as in Definition 7. Let $U \subset X$ be an open subset as in (7.1) and $V \to U$ any projective, toroidal resolution of the pair $(U, \mathrm{red}\, U_k)$. As in [5, Prop. 37] we obtain that $\mathrm{Sk}(V) \subset X^{\mathrm{an}}$ is independent of the choice of V and that it is a subdivision of the dual intersection complex of $\mathrm{red}\, X_k$. We call $\mathrm{Sk}(X) := \mathrm{Sk}(V)$ the *Berkovich skeleton* of X. If $(X, \mathrm{red}\, X_k)$ is toroidal, then we can take $U = X$ and V is an snc-model of X°. In particular, $\mathrm{Sk}(X)$ is a strong deformation retract of X^{an}.

Theorem 24 *Let X° be a projective smooth K-variety with semi-ample canonical class. Let X be a projective,* qdlt *model of X° over R such that $K_X + \operatorname{red} X_k$ is semi-ample.*

(1) *The Berkovich skeleton* $\operatorname{Sk}(X)$ *coincides with the essential skeleton* $\operatorname{Sk}(X^\circ)$.

(2) $\operatorname{Sk}(X^\circ)$ *is a strong deformation retract of* X^{an}.

Proof Note that a model X as in the statement exists by Corollary 4.

As in Definition 7.1, let $U \subset X$ be the maximal open subscheme such that $(U, \operatorname{red} U_k)$ is toroidal. Choose a toroidal, projective resolution $V \to U$ and extend it to a projective log resolution $h : Y \to X$ of $(X, \operatorname{red} X_k)$. Thus $\operatorname{Sk}(X) = \operatorname{Sk}(V)$ by definition. Write

$$K_Y + \operatorname{red} Y \sim h^*(K_X + \operatorname{red} X_k) + \textstyle\sum_E a_E E,$$

where the divisors E are h-exceptional. Over U we have a toroidal resolution, hence the h-exceptional divisors that meet V have discrepancy -1 and they are contained in $\operatorname{red} Y$. Thus $h(E) \subset X \setminus U$ whenever $a_E \neq 0$ and hence $a_E > 0$ for every such E since none of the log canonical centers is contained in $X \setminus U$. A straightforward generalization of the arguments in [23, 3.3.2–3] now shows that $\operatorname{Sk}(X)$ coincides with the essential skeleton $\operatorname{Sk}(X^\circ)$, proving (1).

Next we prove (2). Since the essential skeleton $\operatorname{Sk}(X^\circ)$ depends only on the generic fiber, we can choose our minimal, qdlt-model as in Complement 5. That is, there is a sequence of models X^i such that $(X^0, \operatorname{red} X^0_k)$ is toroidal, each $\phi^i : X^i \dashrightarrow X^{i+1}$ is a $(K_{X^i} + \operatorname{red} X^i_k)$-negative contraction or flip and $(X^m, \operatorname{red} X^m_k) = (X, \operatorname{red} X_k)$. We already noted that $\operatorname{Sk}(X^0)$ is a strong deformation retract of X^{an}, so that it suffices to show that $\operatorname{Sk}(X)$ is a strong deformation retract of $\operatorname{Sk}(X^0)$.

Since [5, Prop. 25] is still true in this setting, we can apply [5, Thm. 19] to conclude that the dual complex $\mathrm{D}(X^{i+1}_k)$ is obtained from $\mathrm{D}(X^i_k)$ by removing all the faces corresponding to the strata of $Z \subset X^i_k$ such that ϕ^i is not an open embedding at the generic point of Z. Arguing as in [5, Lem. 21] we conclude that $\operatorname{Sk}(X^i)$ collapses to $\operatorname{Sk}(X^{i+1})$. At the end we obtain that $\operatorname{Sk}(X^0)$ collapses to $\operatorname{Sk}(X^m) = \operatorname{Sk}(X^\circ)$ hence $\operatorname{Sk}(X^\circ)$ is a strong deformation retract of X^{an}. $\square$

Acknowledgements This work was started during the workshop *Collapsing Calabi–Yau Manifolds.* We thank the Simons Center for its hospitality and D. Abramovich for helpful comments and references.

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