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Spherical Tropical Geometry: a Survey of Recent Developments

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Abstract This is a survey of some recent results on spherical tropical geometry.

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1 Introduction

We give a brief survey of some recent results on spherical tropical geometry and spherical Gröbner theory following [20] and [6]. The classical Gröbner theory deals with ideals in a polynomial ring and tropical geometry deals with subvarieties/subschemes in an algebraic torus (or equivalently ideals in a Laurent polynomial ring). We will explain how to extend some key notions and results in these theories replacing an algebraic torus with a spherical homogeneous space. Vogainnou uses spherical tropical varieties to construct the so-called tropical compactifications for subvarieties in a spherical homogeneous space, extending the earlier work of Tevelev for subvarieties in an algebraic torus [18].

As far as the authors know, the idea of developing tropical geometry for spherical varieties goes back to Gary Kennedy [7]. We should also mention the new preprint of Nash [13] which extends the notion of spherical tropical variety for a subvariety in a spherical homogeneous space to a spherical embedding.

One of our motivations for the study of spherical tropical geometry is to give a description of the *ring of conditions* of a spherical homogeneous space in terms of "balanced" fans generalizing the corresponding well known picture in the toric case.

2 Preliminaries on Tropical Geometry

Throughout this survey, $\mathbf{k}$ denotes the base field which we take to be algebraically closed and with characteristic 0. We let $\mathcal{K}$ to be a the field of formal Puiseux series $\mathbf{k}\{\{t\}\}$ over $\mathbf{k}$. It is the algebraic closure of the field of Laurent series $\mathbf{k}((t))$. Recall that each element of $f \in \mathcal{K}$

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is a series of the form $f(t) = \sum_{i=m}^{\infty} a_i t^{i/k}$ for some $m \in \mathbb{Z}$ and $k \in \mathbb{N}$. The field of Puiseux series has a natural valuation $\mathrm{val} : \mathcal{K} \setminus \{0\} \to \mathbb{Q}$ which assigns to each f its order of t, that is, $\mathrm{val}(f) = m/k$ provided that $a_m \neq 0$. More generally, we can take $\mathcal{K}$ to be any (algebraically closed) field extension of $\mathbf{k}$ equipped with a valuation which is trivial on $\mathbf{k}$.

From the point of view of algebraic geometry, tropical geometry is the study of behavior at infinity of subvarieties of an algebraic torus $(\mathbf{k}^*)^n$. Let $Y \subset (\mathbf{k}^*)^n$ be a subvariety. The behavior at infinity of Y is encoded in a polyhedral fan in $\mathbb{Q}^n$, called the *tropical variety* of Y. It consists of the leading exponents of all the formal Puiseux curves which lie on Y. More precisely, let $\mathrm{Trop} : (\mathcal{K}^*)^n \to \mathbb{Q}^n$ be the map defined by

$$\mathrm{Trop}(\gamma) = (\mathrm{val}(\gamma_1), \ldots, \mathrm{val}(\gamma_n)),$$

for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathcal{K}^*)^n$. The tropical variety $\mathrm{Trop}(Y)$ is simply the image of $Y(\mathcal{K})$ under this map, i.e.,

$$\mathrm{Trop}(Y) = \{\mathrm{Trop}(\gamma) \mid \gamma \in Y(\mathcal{K})\}.$$

It is a basic result in tropical geometry that $\mathrm{Trop}(Y)$ is the support of a rational polyhedral fan in $\mathbb{Q}^n$ (see [12, Section 3.3]). More generally, we can take Y to be a subvariety defined over $\mathcal{K}$. In this case, one shows that $\mathrm{Trop}(Y)$ is a polyhedral complex (instead of a polyhedral fan) in $\mathbb{Q}^n$.

It is a natural question how one can describe $\mathrm{Trop}(Y)$ if Y is given as a zero set of an ideal I in the Laurent polynomial algebra $\mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$. The answer to this question is the content of the so-called fundamental theorem of tropical geometry which we now explain. Let $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a Laurent polynomial. Here $x = (x_1, \ldots, x_n)$, $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and we have used multi-index notation $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$. To f we assign its *tropical polynomial* F which is a piecewise linear function given by:

$$F(w) = \min\{w \cdot \alpha \mid c_\alpha \neq 0\}.$$

More generally, we can take $f(x) \in \mathcal{K}[x_1^{\pm}, \ldots, x_n^{\pm}]$, i.e., the coefficients c_α of f can be Puiseux series. In this case $F(x) = \min\{\mathrm{val}(c_\alpha) + (w \cdot \alpha) \mid c_\alpha \neq 0\}$.

The *tropical hypersurface* $V(F)$ associated to f is by definition the set of $w \in \mathbb{Q}^n$ where the above minimum is attained at least twice. Finally, the *tropical variety* $\mathrm{trop}(I)$ associated to an ideal $I \in \mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is

$$\mathrm{trop}(I) = \bigcup_{f \in I} V(F).$$

It is a basic result that in the above intersection only a finite number of the f suffices (see [12, Section 2.6]). But it is not enough to take a generating set for I. A set $\mathcal{T} \subset I$ such that $\mathrm{trop}(I) = \bigcup_{f \in \mathcal{T}} V(F)$, is called a *tropical basis* for I.

Now, let $Y \subset (\mathbf{k}^*)^n$ be a subvariety with ideal $I = I(Y) \in \mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$. The fundamental theorem of tropical geometry asserts that ([12, Section 3.2]):

Theorem 2.1 $\mathrm{Trop}(Y)$ *and* $\mathrm{trop}(I)$ *coincide.*

The proofs of the above results rely on the Gröbner basis theory of ideals in a polynomial ring and in particular on the notion of the Gröbner fan of a homogeneous ideal.

For a polynomial $f = \sum_\alpha c_\alpha x^\alpha \in \mathbf{k}[x_1, \ldots, x_n]$ and a vector $w \in \mathbb{Q}^n$, one defines the initial form $\mathrm{in}_w(f)$ to be $\sum_\beta c_\beta x^\beta$ where the sum is over β for which the minimum $\min\{w \cdot \alpha \mid c_\alpha \neq 0\}$

is attained. Also for an ideal $I \subset \mathbf{k}[x_1, \dots, x_n]$, the initial ideal $\mathrm{in}_w(I)$ is the ideal generated by $\mathrm{in}_w(f)$, for all $f \in I$. Given an ideal I one can group together the vectors in $\mathbb{Q}^n$ by saying that $w_1 \sim w_2$, $w_1, w_2 \in \mathbb{Q}^n$, if $\mathrm{in}_{w_1}(I) = \mathrm{in}_{w_2}(I)$. The following is a basic theorem in Gröbner theory (see [16, Section 2]).

Theorem 2.2 (Gröbner fan) *Let I be a homogeneous ideal (with respect to a positive weighting of the variables $x_1, \dots, x_n$). Then the closures of equivalence classes of $\sim$ are rational convex polyhedral cones and form a complete fan in $\mathbb{Q}^n$.*

Finally, it is worthwhile to mention that there is yet another description of the tropical variety of Y in terms of toric varieties. For $w \in \mathbb{Q}^n$, let X_w be the toric variety associated to the fan with single ray generated by w. The variety X_w contains a unique divisor D_w at infinity. One then has: $w \in \mathrm{Trop}(Y)$ if and only if $\overline{Y} \cap D_w \neq \emptyset$, where $\overline{Y}$ denotes the closure of Y in X_w.

This last result motivates the important notion of a tropical compactification. Let $Y \subset (\mathbf{k}^*)^n$ be a subvariety. The closure $\overline{Y}$ in a toric variety $X_\Sigma \subset (\mathbf{k}^*)^n$ is a tropical compactification if:

(1) $\overline{Y}$ is a complete variety.

(2) The multiplication map $(\mathbf{k}^*)^n \times \overline{Y} \to X$ is faithfully flat. In particular, this means that $\overline{Y}$ intersects all the torus orbits in X_Σ.

Note that in the above, the toric variety X_Σ itself need not be complete.

Theorem 2.3 (Tevelev) *There exists a tropical compactification. It corresponds to the toric variety associated to a fan whose support is* $\mathrm{Trop}(Y)$*.*

The notion of a tropical compactification appears in [18], although the concept of a "good" compactification for a subvariety has been considered previously by several authors. De Concini and Procesi prove its existence in a very general setting and without referring to any tropical varieties [2].

3 Preliminaries on Spherical Varieties

Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$. We let T, B and U be a maximal torus in G, a Borel subgroup containing T and a maximal unipotent subgroup contained in B respectively. We denote the weight lattice and the semigroup of dominant weights of G by Λ and Λ^+ respectively. We also denote the positive Weyl chamber (corresponding to a choice of Borel B) by $\Lambda^+_{\mathbb{R}}$.

A normal G-variety X is called *spherical* if B has a dense open orbit (note that since all the Borel subgroups are conjugate this is independent of the choice of B). A homogeneous space X is spherical if it is spherical for the left action of G. It is well known that a quasi-projective G-variety X is spherical if and only if for any G-linearized line bundle L on X the space of sections $H^0(X, L)$ is a multiplicity-free G-module, i.e., each irreducible G-module appears with multiplicity at most 1.

We let $\Lambda_X \subset \Lambda$ denote the lattice of weights of B-eigenfunctions in the field of rational functions $\mathbf{k}(X)$. Since B has an open orbit it follows that the map which assigns to a B-eigenfunction its weight, gives an isomorphism between $\mathbf{k}(X)^{(B)}/\mathbf{k}^*$ and Λ_X. One can show that there is a natural choice of a maximal torus $T \subset G$ such that the weight lattice of $T_X = T/(T \cap H)$ can be identified with the lattice Λ_X. The lattice Λ_X is usually called the *weight* *lattice of X.*

Next important object associated to X is the valuation cone $\mathcal{V}_X$. It is the set of all G-invariant valuations $v : \mathbf{k}(X) \setminus \{0\} \to \mathbb{Q}$. Evaluating a G-invariant valuation on the set of B-eigenfunctions in $\mathbf{k}(X)$ gives us a homomorphism from $\mathcal{V}_X$ to the $\mathbb{Q}$-vector space $\mathrm{Hom}(\Lambda_X, \mathbb{Q})$. It is well known that this homomorphism is one-to-one. It is a fundamental result of Brion [1] and Knop [8] that the image of $\mathcal{V}_X$ in $\mathrm{Hom}(\Lambda_X, \mathbb{Q})$ is a cosimplicial cone which is the fundamental domain for an action of a finite reflection group. The reflections are with respect to the hyperplanes orthogonal to the so-called *spherical roots* of X. The set of invariant valuations $\mathcal{V}_X$ hence is referred to as the *valuation cone* of X. It is a main object in the Luna–Vust classification of spherical embeddings by colored fans [10, 11]. It plays a role analogous to the vector space $N_{\mathbb{Q}}$ of one-parameter subgroups in the classification of toric varieties by fans.

4 Spherical Tropicalization

Let $X = G/H$ be a spherical homogeneous space and $Y \subset X$ a subvariety. Following [20] we explain how to associate to Y a fan in the valuation cone $\mathcal{V}_X$ which encodes the "behavior at infinity" of Y in regard to G-equivariant embeddings of G/H. We start by recalling a classical result of Sumihiro [17].

Theorem 4.1 (Sumihiro) *Let $v : \mathbf{k}(X) \to \mathbb{R} \cup \{\infty\}$ be a valuation.*

(1) *For every $0 \neq f \in \mathbf{k}(X)$, there exists a nonempty Zariski open subset $U_f \subset G$ such that the value $v(g \cdot f)$ is the same for all $g \in U_f$. Let us denote this value by $\bar{v}(f)$, i.e.,*

$$\bar{v}(f) = v(g \cdot f), \quad \forall g \in U_f.$$

(2) *We have $\bar{v}(f) = \min\{v(g \cdot f) \mid g \in G\}$.*

(3) *$\bar{v}$ is a G-invariant valuation on X.*

Now let $\gamma \in X(\mathcal{K})$ be a (formal) curve on X, i.e., a point of X defined over the field $\mathcal{K}$ of Puiseux series. Theorem 4.1 implies that $\bar{v}_\gamma$ defined below is a well-defined G-invariant valuation:

$$\bar{v}_\gamma(f) = \mathrm{val}(f(g \cdot \gamma(t))),$$

for every $0 \neq f \in A$ and $g \in G$ in general position. Considering valuations of the form $\bar{v}_\gamma$ goes back to [11]. Following [20] we call the map

$$\mathrm{Trop} : X(\mathcal{K}) \to \mathcal{V}_X, \quad \gamma \mapsto \bar{v}_\gamma, \tag{4.1}$$

the *spherical tropicalization map.* The *spherical tropical variety* $\mathrm{Trop}(Y)$ of Y is the image of $Y(\mathcal{K})$ under the map Trop. In [20] the following is proved:

Theorem 4.2 (Vogiannou) $\mathrm{Trop}(Y)$ *is the support of a rational polyhedral fan in the valuation cone $\mathcal{V}_X$. Moreover, there is a fan Σ with support* $\mathrm{Trop}(Y)$ *such that the corresponding spherical embedding gives a tropical compactification of Y.* (*Here we mean the spherical embedding associated to Σ in the sense of Luna–Vust and the fan Σ has no colors, i.e., the corresponding embedding is toroidal.*)

The proof of the above theorem in [20] relies on the Luna–Vust theory of spherical embeddings.

Remark 4.3 We would like to point out that the spherical tropicalization map can also be constructed using the non-Archimedean spherical Cartan decomposition in [11] (see also [3, 15] for the non-Archimedean spherical Cartan decomposition).

Below is a baby example to illustrate the construction.

Example 4.4 Consider the variety $X = \mathbb{A}^2 \setminus \{(0,0)\}$ for the natural action of $G = \mathrm{SL}(2, \mathbf{k})$. It is a spherical homogeneous space. The algebra of regular functions $\mathbf{k}[X]$ is just the polynomial ring $\mathbf{k}[x, y]$. The weight lattice Λ_X coincides with the weight lattice Λ of G and can be identified with $\mathbb{Z}$. The function $f(x, y) = y$ is a B-eigenfunction in $\mathbf{k}[X]$ whose weight is 1, namely the generator of Λ_X. Let $\gamma = (\gamma_1, \gamma_2)$ be a formal curve in $X = \mathbb{A}^2 \setminus \{0\}$. Let us write $\gamma_1(t) = \sum_i a_i t^i$ and $\gamma_2(t) = \sum_i b_i t^i$. Let $g = \left[\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right]$. We compute that $f(g \cdot \gamma(t)) = g_{21}\gamma_1 + g_{22}\gamma_2$. From the definition we have $\bar{v}_\gamma(y) = \mathrm{val}(g \cdot \gamma(t))$ for g in general position. Thus

$$\bar{v}_\gamma(y) = \mathrm{val}(g_{21}\gamma_1(t) + g_{22}\gamma_2(t)) = \min(\mathrm{val}(\gamma_1(t)), \mathrm{val}(\gamma_2(t))). \tag{4.2}$$

It is relatively straightforward to extend the above notion of tropicalization to the context of Berkovich spaces. We briefly explain this following [6, Section 4.6]. First we recall the notion of a Berkovich analytic space or analytification of a variety X. It plays a central role in non-Archimdean geometry.

Let A be a finitely generated $\mathbf{k}$-algebra and $X = \mathrm{Spec}(A)$ the corresponding affine variety. Let $\tilde{A} = A \otimes_{\mathbf{k}} \mathcal{K}$ where as before $\mathcal{K} = \mathbf{k}\{\{t\}\}$ is the field of formal Puiseux series in an indeterminate t. It is equipped with a natural valuation $\mathrm{val} : \mathcal{K} \to \mathbb{Q} \cup \{\infty\}$.

Definition 4.5 (Multiplicative seminorm) *A function $p : \tilde{A} \to \mathbb{R}_{\geq 0}$ is called a multiplicative seminorm on A if it satisfies the following*:

(a) $p(fg) = p(f)p(g)$,

(b) $p(\lambda) = \exp(-\mathrm{val}(\lambda))$,

(c) $p(f + g) \leq \max(p(f), p(g))$, *for all $f, g \in \tilde{A}$ and $\lambda \in \mathcal{K}$.*

The analytification X^{an} of X is the collection of all multiplicative seminorms on $\tilde{A}$. We endow X^{an} with the coarsest topology in which the maps $X^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(f)$, are continuous for every $f \in \tilde{A}$.

For a multiplicative seminorm p, one defines the corresponding valuation $v_p : \tilde{A} \to \mathbb{Q} \cup \{\infty\}$ by $v_p(f) = -\log(p(f))$, for all $f \in \tilde{A}$. (In this context, it is more convenient to consider a valuation as a map from $\tilde{A}$ to $\mathbb{R} \cup \{\infty\}$ and define the value of 0 to be ∞.)

There is a natural embedding $j : X(\mathcal{K}) \hookrightarrow X^{\mathrm{an}}$ given by restricting to points in $X(\mathcal{K})$. More precisely, for each point $\gamma \in X(\mathcal{K})$, we let $p = j(\gamma)$ to be the multiplicative seminorm defined by:

$$j(\gamma)(f) = \exp(-\mathrm{val}(f(\gamma))). \tag{4.3}$$

Now let X be a quasi-affine spherical homogeneous space with ring of regular functions $A = \mathbf{k}[X]$. Recall that to any valuation v on X we can associate a G-invariant valuation $\bar{v}$ on X (see Theorem 4.1). For any $f \in \mathbf{k}(X)$, the value $\bar{v}(f)$ is defined by:

$$\bar{v}(f) = v(g \cdot f),$$

for any $g \in G$ in general position, i.e., g lies in some Zariski open subset U_f of G.

More generally, let $Y \subset X$ be a subvariety. Let $\pi : A \to \mathbf{k}[Y]$ be the algebra homomorphism induced by the inclusion of Y in X.

For a valuation $v : \mathbf{k}[Y] \to \mathbb{R} \cup \{\infty\}$, we denote by $\bar{v} : A \to \mathbb{R} \cup \{\infty\}$ the valuation defined as follows. For any $f \in A$ let:

$$\bar{v}(f) = v(\pi(g \cdot f)), \tag{4.4}$$

for g in some Zariski open subset U_f. Now let $p \in Y^{\mathrm{an}}$ with the associated valuation v_p. Let $\bar{v}_p$ be the G-invariant valuation on $\mathbf{k}(X)$ associated to v_p as above.

Definition 4.6 (Spherical tropicalization map) *We define the spherical tropicalization map* $\mathbf{Trop} : Y^{\mathrm{an}} \to \mathcal{V}_X$ *by*:

$$p \mapsto \bar{v}_p.$$

Proposition 4.7 *We have the following*:

(1) *The map* $\mathbf{Trop} : Y^{\mathrm{an}} \to \mathcal{V}_X$ *is continuous.*

(2) *The map* $\mathbf{Trop}$ *extends the tropicalization map* $\mathrm{Trop} : Y(\mathcal{K}) \to \mathcal{V}_X$ *introduced in* (4.1). *That is, the diagram below commutes*:

$$\begin{tikzcd} Y(\mathcal{K}) \arrow[rr, hook, "j"] \arrow[dr, "\mathrm{Trop}"'] & & Y^{\mathrm{an}} \arrow[dl, "\mathbf{Trop}"] \\ & \mathcal{V}_X & \end{tikzcd}$$

5 Gröbner Theory for G-algebras

The purpose of this section is to discuss an extension of the Gröbner theory for ideals in a polynomial ring to ideals in the coordinate ring of an affine spherical G-variety. As in the usual tropical geometry, this can be used to give a description of the spherical tropical variety of a subvariety $Y \subset G/H$ in terms of its associated ideal.

Throughout this section, X is an affine spherical variety with $A = \mathbf{k}[X]$ its coordinate ring. We write

$$A = \bigoplus_{\lambda \in \Lambda^+} W_\lambda,$$

for the isotypic decomposition of A as a G-module, i.e., W_λ is the sum of all copies of the irreducible G-module V_λ in A (since X is spherical we know $W_\lambda = \{0\}$ or V_λ).

Analogous to the well known notion of dominant weight order on Λ, one defines a partial order $>_X$ in Λ_X which we call the *spherical dominant order*. For $\lambda, \mu \in \Lambda_X$, we say that $\lambda >_X \mu$ is $\mu - \lambda$ is a linear combination of spherical roots with non-negative integer coefficients.

We take a total order $\succ$ on Λ_X that refines the spherical dominant order $>_X$ and is maximum well-ordered on the semigroup Λ_A^+ of highest weights in A. We remark that if A is a positively graded G-algebra and each graded piece is finite dimensional (e.g., A is the homogeneous coordinate ring of a spherical projective G-variety), then A always has such a total order $\succ$ regarded as a $(G \times \mathbf{k}^*)$-algebra.

The total order $\succ$ gives rise to a filtration on A. Namely, for each $\lambda \in \Lambda^+$ we define:

$$A_{\succeq\lambda} = \bigoplus_{\mu \succeq \lambda} W_\mu.$$

The space $A_{\succ\lambda}$ is defined similarly. We denote the associated graded of this filtration by $\mathrm{gr}_\succ(A)$, that is:

$$\mathrm{gr}_\succ(A) = \bigoplus_{\lambda \in \Lambda^+} A_{\succeq\lambda}/A_{\succ\lambda}.$$

In fact, one shows that, as a G-algebra, $\mathrm{gr}_{\succ}(A)$ is isomorphic to A_{hc}, the horospherical contraction of A. We recall that the horospherical contraction A_{hc} is a G-algebra that is isomorphic to A as a G-module, but its ring structure has the following property: for any two dominant weights λ, μ, the product of the λ-isotypic and μ-isotypic components lies in the $(\lambda+\mu)$-isotypic component (see [14]).

For an ideal $I \subset A$, we define its initial ideal $\mathrm{in}_{\succ}(I)$ to be the ideal in $\mathrm{gr}_{\succ}(A)$ generated by the images of all $f \in I$. We say that a subset $\mathcal{G} \subset I$ is a *spherical Gröbner basis* if the image of $\mathcal{G}$ in $\mathrm{in}_{\succ}(I)$ generates this ideal. In [6] the authors give a generalization of the well known division algorithm to this setting and prove that a spherical Gröbner basis $\mathcal{G}$ is a set of ideal generators for the original ideal I. Generalizing an analogous statement from Gröbner theory of ideals in a polynomial ring, in [6] the following is proved.

Theorem 5.1 *An ideal $I \subset A$ has a finite number of initial ideals (regarded as ideals in the horospherical contraction A_{hc} of A).*

This theorem then implies the existence of a universal spherical Gröbner basis for I.

Next we consider generalizations of the notions of initial ideal with respect to a vector $w \in \mathbb{Q}^n$ and Gröbner fan of an ideal. In the spherical Gröbner theory the role of a vector $w \in \mathbb{Q}^n$ is played by a G-invariant valuation on the field of rational functions $\mathbf{k}(X)$. A G-invariant valuation v gives rise to a filtration on the algebra A defined as follows. For every $a \in \mathbb{Q}$ put:

$$A_{v \geq a} = \{f \in A \mid v(f) \geq a\},$$

(the subspace $A_{v>a}$ is defined similarly). We note that $A_{v\geq a} = \bigoplus_{\langle v,\gamma\rangle \geq a} W_\gamma$. The associated graded algebra of v is:

$$\mathrm{gr}_v(A) = \bigoplus_{a\in\mathbb{Q}} A_{v\geq a}/A_{v>a}.$$

Since v is G-invariant, each subspace in the filtration is G-stable and the algebra $\mathrm{gr}_v(A)$ is naturally a G-algebra. We make the following observation:

Proposition 5.2 *For $v \in \mathcal{V}_X$, the associated graded algebra $\mathrm{gr}_v(A)$ depends only on the face σ of the cone $\mathcal{V}_X$ which contains v in its relative interior (thus we also write $\mathrm{gr}_\sigma(A)$ instead of $\mathrm{gr}_v(A)$). When v lies in the interior of $\mathcal{V}_X$ then $\mathrm{gr}_v(A)$ is isomorphic to the horospherical contraction A_{hc}.*

For $f \in A$, we let $\mathrm{in}_v(f)$ denote the image of f in the quotient space $A_{v\geq a}/A_{v>a}$, where $a = v(f)$. For an ideal $I \subset A$, the initial ideal $\mathrm{in}_v(I) \subset \mathrm{gr}_v(A)$ is the ideal generated by $\mathrm{in}_v(f)$, for all $f \in I$. Extending the usual Gröbner theory we define an equivalence relation on the valuations: for $v_1, v_2 \in \mathcal{V}_X$ we say $v_1 \sim v_2$ if they lie on the relative interior of the same face σ of $\mathcal{V}_X$ and also $\mathrm{in}_{v_1}(I) = \mathrm{in}_{v_2}(I)$ regarded as ideals in $\mathrm{gr}_\sigma(A)$. We would like to show that the equivalence classes of $\sim$ form a fan. For this, we need to assume that $A = \bigoplus_{i\geq 0} A_i$ is graded and G acts on A preserving the grading. Moreover, we assume that each graded piece A_i is a multiplicity free G-module. This implies that A is a multiplicity free $(G \times \mathbf{k}^*)$-algebra. Thus A is the ring of regular functions on a spherical $(G \times \mathbf{k}^*)$-variety X. In this situation by the valuation cone $\mathcal{V}_X$ we mean the cone of $(G \times \mathbf{k}^*)$-invariant valuations.

Theorem 5.3 *Let A be graded as above. Let $I \subset A$ be a homogeneous ideal. Then the closures of equivalence classes of $\sim$ form a fan which we call the spherical Gröbner fan of I.*

There are some important differences between the toric/polynomial case and the general spherical case which makes the spherical theory more complicated.

• In the torus case, the isotypic components are 1-dimensional corresponding to different (Laurent) monomials, while in the general spherical case they are irreducible G-modules and usually have dimension greater than 1.

• If $f_\alpha = x^\alpha$, $f_\beta = x^\beta$ are two monomials in a polynomial algebra, then $f_\alpha f_\beta = f_{\alpha+\beta}$. In the spherical case, if $f_\gamma \in W_\gamma$, $f_\mu \in W_\mu$ where $A = \bigoplus_\lambda W_\lambda$ is the isotypic decomposition of the G-algebra A, then in general, $f_\gamma f_\mu$ does not necessarily lie in $W_{\gamma+\mu}$ but rather in $W_{\gamma+\mu}$ direct sum with W_λ where λ is greater than $\gamma + \mu$ in the spherical dominant order $>_X$.

6 Spherical Tropical Variety from an Ideal

In this section following [6], we describe how to define a spherical tropical variety for a subvariety in the open Borel orbit using its defining ideal. We then state a spherical version of the fundamental theorem of tropical geometry.

To this end, we apply the spherical Gröbner theory results (specifically the spherical Gröbner fan) from the previous section to the algebra of sections of a line bundle on a spherical homogeneous space.

Let $X = G/H$ be a spherical homogeneous space (we do not require X to be quasi-affine). Let $v \in \mathcal{V}_X$ be a G-invariant valuation. Let X_v denote the equivariant embedding of X corresponding to the fan with a single ray generated by v. The spherical variety X_v consists of two G-orbits: the open G-orbit X and a G-stable prime divisor which we denote by D_v. The orbit D_v is the unique closed G-orbit in X_v. The geometric valuation corresponding to the divisor D_v is a multiple of the valuation v.

We fix a Borel subgroup B. We let X_B be the open B-orbit in $X = G/H$. One knows that X_B is an affine variety. We denote the set of B-stable prime divisors in X by $\mathcal{D}(X)$. One observes that $X_B = X \setminus \bigcup_{D\in\mathcal{D}(X)} D$. We also denote the open B-orbit in D_v by D'_v, and the set of B-stable prime divisors in X_v by $\mathcal{D}(X_v)$. One defines a subvariety $X_{v,B} \subset X_v$ by:

$$X_{v,B} = X_v \Big\backslash \bigcup_{D\in\mathcal{D}(X_v)\setminus\{D_v\}} D.$$

One shows the following (see [10, Theorem 2.1]): $X_{v,B}$ is a B-stable affine subvariety of X_v and $X_{v,B} \cap D_v$ is the B-orbit D'_v. Moreover, the coordinate ring of $X_{v,B}$ can be described as:

$$\mathbf{k}[X_{v,B}] = \{f \in \mathbf{k}[X_B] \mid v(f) \geq 0\}.$$

More generally, one can find a B-stable affine neighborhood of any closed G-orbit in a spherical embedding (in [10, Section 2] this affine neighborhood is denoted by X_0).

The notions of associated graded and initial ideal for $\mathbf{k}[X_B]$ with respect to a valuation $v \in \mathcal{V}_X$ are defined as before. (Notice that we are not assuming here that X is quasi-affine and hence $\mathbf{k}[X]$ could be very small, e.g., it could consist only of constant functions, while on the other hand, X_B is always an affine variety.) More precisely, for every $a \in \mathbb{Q}$, we put $\mathbf{k}[X_B]_{v\geq a} = \{f \in \mathbf{k}[X_B] \mid v(f) \geq a\}$ (the subspace $\mathbf{k}[X_B]_{v>a}$ is defined similarly). The corresponding associated graded is:

$$\mathrm{gr}_v(\mathbf{k}[X_B]) = \bigoplus_{a\in\mathbb{Q}} \mathbf{k}[X_B]_{v\geq a}/\mathbf{k}[X_B]_{v>a}.$$

We note that each subspace in the filtration is B-stable and $\mathrm{gr}_v(\mathbf{k}[X_B])$ is a B-algebra. One can show that if v_1, v_2 lie in the relative interior of the same face σ of the valuation cone $\mathcal{V}_X$, then the corresponding associated graded algebras $\mathrm{gr}_{v_1}(\mathbf{k}[X_B])$ and $\mathrm{gr}_{v_2}(\mathbf{k}[X_B])$ are naturally isomorphic.

For each f with $v(f) = a$, let $\mathrm{in}_v(f)$ denote the image of f in the quotient space $\mathbf{k}[X_B]_{v\geq a}/\mathbf{k}[X_B]_{v>a}$. For an ideal $J \subset \mathbf{k}[X_B]$, we let $\mathrm{in}_v(J)$ be the ideal in $\mathrm{gr}_v(\mathbf{k}[X_B])$ generated by all the $\mathrm{in}_v(f)$, $\forall f \in J$.

Now we define the notion of a spherical tropical variety of a subscheme of X_B given by an ideal in the coordinate ring $\mathbf{k}[X_B]$).

Definition 6.1 (Spherical tropical variety of an ideal in the coordinate ring of X_B) *Let $J \subset \mathbf{k}[X_B]$ be an ideal. We define* $\mathrm{trop}(J)$ *to be the set of all $v \in \mathcal{V}_X$ such that the initial ideal* $\mathrm{in}_v(J)$ *is not equal to* $\mathrm{gr}_v(\mathbf{k}[X_B])$. *In other words,* $\mathrm{in}_v(J)$ *does not contain a unit element. We call* $\mathrm{trop}(J)$ *the spherical tropical variety of J.*

Theorem 6.2 (Fan structure) *For every ideal $J \subset \mathbf{k}[X_B]$, the spherical tropical variety* $\mathrm{trop}_B(J)$ *is the support of a rational polyhedral fan.*

We say that a set $\mathcal{T} = \mathcal{T}(J) \subset J$ is a *spherical tropical basis* for J if for every $v \in \mathcal{V}_X$ the following holds: $\mathrm{in}_v(J) \subset \mathrm{gr}_v(\mathbf{k}[X_B])$ contains a unit element, i.e., $v \notin \mathrm{trop}(J)$, if and only if there exists $f \in \mathcal{T}$ such that $\mathrm{in}_v(f)$ is a unit element. The following is prove in [6, Section 4.3]

Theorem 6.3 (Existence of a finite spherical tropical basis) *Every ideal $J \subset \mathbf{k}[X_B]$ has a finite spherical tropical basis.*

Let us say a few words about how spherical Gröbner theory is used in [6] to prove the above statements. This is a generalization of the usual approach in tropical geometry. Namely, given an ideal $I \subset \mathbf{k}[x_1^{\pm}, \ldots, x_n^{\pm}]$ in a Laurent polynomial ring, one considers its *homogenization* $\tilde{I} \subset \mathbf{k}[x_1, \ldots, x_{n+1}]$ and relates the tropical variety of I with the Gröbner fan of the homogeneous ideal $\tilde{I}$. To extend this to the spherical setting, we consider a very ample G-line bundle on a projective spherical embedding $\bar{X}$ of G/H. Now consider the ring of sections A of L, that is the graded algebra:

$$A = \bigoplus_{i\geq 0} H^0(\bar{X}, L^{\otimes i}),$$

where we set $A_0 = \mathbf{k}$. Since $\bar{X}$ is spherical for the action of G, one sees that A is a multiplicity-free $(G \times \mathbf{k}^*)$-algebra. We assume that there is a B-weight section $s \in H^0(\bar{X}, L)$ which vanishes on all B-stable divisors in X. Given such a section s one can define a homomorphism $\pi : A \to \mathbf{k}[X_B]$ by sending $f \in A_i$ to $f/s^i \in \mathbf{k}[X_B]$, for all i. For an ideal $J \subset \mathbf{k}[X_B]$, we define the homogeneous ideal $\tilde{J}$ to be the ideal generated by homogeneous elements of $\pi^{-1}(J) \subset A$ (we think of this as a generalization of the notion of homogenization of an ideal in a Laurent polynomial ring). We then use the existence of spherical Gröbner fan for the homogeneous ideal $\tilde{J}$ in the multiplicity-free $(G \times \mathbf{k}^*)$-algebra A to conclude the existence of a fan structure on $\mathrm{trop}_B(J)$ and a finite tropical basis for J.

Finally we give a generalization of the fundamental theorem of tropical geometry to the spherical setting. It states that the spherical tropical variety as defined using initial ideals in Borel charts coincides with the spherical tropical variety defined using tropicalization map and invariant valuations.

Let $Y \subset X = G/H$ be a subvariety. For each Borel subgroup B, we let J_B be the defining ideal of Y intersected with the open B-orbit X_B. The following is proved in [6, Section 4.5]

Theorem 6.4 (Fundamental theorem) *The following coincide*:

(a) *The set* $\mathrm{trop}(Y) = \bigcup_B \mathrm{trop}(J_B)$, *where the union is over all Borel subgroups of* G (*one shows that it is enough to take the union over a finite collection of Borel subgroups*).

(b) *The set* $\mathrm{Trop}(Y) = \{\mathrm{Trop}(\gamma) \in \mathcal{V}_X \mid \gamma \in Y(\mathcal{K})$ is a formal Puiseux curve on $Y\}$.

Example 6.5 As in Example 4.4 consider the spherical variety $X = \mathbb{A}^2 \setminus \{(0,0)\}$ for the natural action of $G = \mathrm{SL}(2, \mathbf{k})$. We recall that this action is transitive. The stabilizer of the point $(0,1)$ is the subgroup U^- of lower triangular matrices with 1's on the diagonal and we identify X with G/U^-. Let B and B^- denote the Borel subgroups of upper triangular and lower triangular matrices respectively. It is easy to see that the B-orbit and B^--orbit of the point $(0,1)$ are $X_B = \mathbb{A}^2 \setminus \{y = 0\}$ and $X_{B^-} = \mathbb{A}^2 \setminus \{x = 0\}$. Thus the coordinate rings $\mathbf{k}[X_B]$ and $\mathbf{k}[X_{B^-}]$ are $\mathbf{k}[x, y, y^{-1}]$ and $\mathbf{k}[x, y, x^{-1}]$ respectively.

Clearly the action of G on X extends to the whole projective plane $\mathbb{P}^2$. One can verify that every G-orbit $O \subset \mathbb{P}^2$ is covered by the open B-orbit and the open B^--orbit contained in O.

A description of the tropicalization of a curve for this example is obtained in [20, Example 3.10]. Namely, $\mathrm{Trop}(Y)$ is the whole $\mathcal{V}_X = \mathbb{Q}$ if Y passes through the origin and it is the negative ray in $\mathbb{Q}$ otherwise. We verify the fundamental theorem (Theorem 6.4) by computing the tropical variety of a curve Y defined by a principal ideal $I \subset \mathbf{k}[x, y]$ (Definition 6.1). Let $I = \langle f \rangle$ be a principal ideal where f is a nonconstant polynomial. We write $f = \sum_{i=m}^{d} f_i$ as the sum of its homogeneous component, i.e., f_i is a homogeneous polynomial of degree i in $\mathbf{k}[x, y]$ and f_m, f_d are nonzero. First consider the case $v \geq 0$. One can check that $\mathrm{in}_{v,B}(f)$ is a unit if f_m is either constant or a power of y. Thus $v \in \mathrm{trop}(I_B)$ if and only if f_m is neither constant nor a power of y. Similarly, $v \in \mathrm{trop}(I_{B^-})$ if and only if f_m is neither constant nor a power of x. Putting these together we see that $v \in \mathrm{trop}(Y)$ if and only if f_m is not a constant. The case $v < 0$ can be dealt with in a similar fashion. In this case we have $v \in \mathrm{trop}(I_B)$ if and only if f_d is not a power of y, and $v \in \mathrm{trop}(I_{B^-})$ if and only if f_d is not a power of x. It thus follows that $v < 0$ always lies in $\mathrm{trop}(Y) = \mathrm{trop}(I_B) \cup \mathrm{trop}(I_{B^-})$. In summary,

$$\mathrm{trop}(Y) = \begin{cases} \mathcal{V}_X, & f_0 = 0, \\ \{v \in \mathcal{V}_X \mid v \leq 0\}, & f_0 \neq 0, \end{cases}$$

as expected.

7 Spherical Amoebas

Finally, we address the notion of amoeba of a variety. When $\mathbf{k} = \mathbb{C}$ one defines a logarithm map on the torus $(\mathbb{C}^*)^n$ as follows. Fix a real number $t > 0$. The logarithm map $\mathrm{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}^n$ is simply defined by:

$$\mathrm{Log}_t(z_1, \ldots, z_n) = (\log_t(|z_1|), \ldots, \log_t(|z_n|)).$$

The amoeba of a subvariety $Y \subset (\mathbb{C}^*)^n$ is defined to be the image of Y under the logarithm map. A well known theorem states that as t approaches 0, the amoeba of Y approaches, in Hausdorff metric, to the tropical variety of Y.

In [6, Section 6] an extension of this notion is suggested for a spherical homogeneous space $X = G/H$. Although for this to work, one needs to assume that (Archimedean) Cartan decomposition holds for X in the following sense. Throughout this section the ground field is $\mathbf{k} = \mathbb{C}$.

Let T_X be the torus associated to $X = G/H$. It can be identified with $T/T \cap H$ for a maximal torus $T \subset G$. Thus T_X can be also identified with the T-orbit of $eH \in X$. The lattice of characters of T_X is Λ_X. We consider the exponential map $\exp : \mathrm{Lie}(T_X) \to T_X \subset X$. The valuation cone sits in the vector space $\mathrm{Hom}(\Lambda_X, \mathbb{Q})$ which in turn we consider as a subset of $\mathrm{Lie}(T_X)$. The image $\exp(\mathcal{V}_X)$ of the valuation cone thus naturally sits in $T_X \subset X$.

Assumption 7.1 ((Archimedean) Cartan decomposition for a spherical homogeneous space) There exists a maximal compact subgroup K of G which is a real algebraic subgroup such that each K-orbit in G/H intersects the image of the valuation cone $\exp(\mathcal{V}_X)$ at a unique point.

In fact, the authors originally conjectured that the above (Archimdean) Cartan decomposition should hold for any spherical homogeneous space. Later, we learned that Victor Batyrev had made the same conjecture some years ago (some related results can be found in [9]).

We can then define the map $\mathbf{L}_t : X \to \mathcal{V}_X$ by:

$$x \mapsto \mathrm{Log}_t((K \cdot x) \cap \exp(\mathcal{V}_X)) \in \mathcal{V}_X,$$

that is, first we intersect the orbit $K \cdot x$ with $\exp(V_X)$ and then map it to the valuation cone by the logarithm map Log_t. We call $\mathbf{L}_t$ a *spherical logarithm map.*

Definition 7.2 (Spherical amoeba) *Let $Y \subset X$ be a subvariety. We denote the image of Y under the map $\mathbf{L}_t$ by $\mathcal{A}_t(Y)$ and call it the spherical amoeba of the subvariety Y.*

In the special case of $X = \mathrm{GL}(n, \mathbb{C})$ regarded as a $(\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}))$-spherical homogeneous space, the general statement that spherical amoeba should approach the spherical tropical variety specializes to a linear algebra statement that the Smith normal form of a matrix (whose entries are Laurent series in one variable t) is a limit of the logarithm of singular values of the matrix as t approaches 0. The authors are not aware of such a statement in the literature relating singular values and invariant factors.

Let us explain this in more detail. Recall that if A is an $n \times n$ complex matrix, the singular value decomposition states that A can be written as:

$$A = U_1 D U_2,$$

where U_1, U_2 are $n \times n$ unitary matrices and D is diagonal with nonnegative real entries. In fact, the diagonal entries of d are the eigenvalues of the positive semi-definite matrix $\sqrt{AA^*}$ where $A^* = \bar{A}^t$. The diagonal entries of D are usually referred to as the *singular values of A*.

Let $A(t)$ be an $n \times n$ matrix whose entires $A_{ij}(t)$ are Laurent series in t (over $\mathbb{C}$). We recall that the Smith normal form theorem (over the ring of formal power series which is a PID) states that $A(t)$ can be written as:

$$A_1(t)\tau(t)A_2(t),$$

where $A_1(t)$, $A_2(t)$ are $n \times n$ matrices with power series entries and invertible over the power series ring, and $\tau(t)$ is a diagonal matrix of the form $\tau(t) = \mathrm{diag}(t^{v_1}, \ldots, t^{v_n})$ for integers $v_1, \ldots, v_n$. The integers $v_1, \ldots, v_n$ are usually called the *invariant factors of $A(t)$*.

We have the following statement ([6, Section 6]). A direct proof can be given using the Hilbert–Courant min-max principle.

Theorem 7.3 *Let $A(t)$ be an $n \times n$ matrix whose entries A_{ij} are Laurent series in t with nonzero radii of convergence. For sufficiently small $t \neq 0$, let $d_1(t) \leq \cdots \leq d_n(t)$ denote the singular values of $A(t)$ ordered increasingly. Also let $v_1 \geq \cdots \geq v_n$ be the invariant factors of $A(t)$ ordered decreasingly. We then have*

$$\lim_{t \to 0}(\log_t(d_1(t)), \ldots, \log_t(d_n(t))) = (v_1, \ldots, v_n).$$

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References

[1] Brion, M.: Vers une généralisation des espaces symétriques. *J. Algebra*, **134**, 115–143 (1990)

[2] De Concini, C., Procesi, C.: Complete symmetric varieties. II. Intersection theory. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 481–513, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985

[3] Gaitsgory, D., Nadler, D.: Spherical varieties and Langlands duality. *Mosc. Math. J.*, **10**(1), 65–137, 271 (2010)

[4] Gelfand, I. M., Kapranov, M. M., Zelevinsky, A. V.: Discriminants, resultants and multidimensional determinants. Reprint of the 1994 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008

[5] Gubler, W.: A guide to tropicalizations. Algebraic and combinatorial aspects of tropical geometry, 125–189, Contemp. Math., 589, Amer. Math. Soc., Providence, RI, 2013

[6] Kaveh, K., Manon, C.: Gröbner theory and tropical geometry on spherical varieties. arXiv:1611.01841

[7] Kennedy, G.: http://u.osu.edu/kennedy.28/files/2014/11/sphericaltropical2-12rpr9e.pdf

[8] Knop, F.: The asymptotic behavior of invariant collective motion. *Invent. Math.*, **116** (1994)

[9] Knop, F., Krötz, B., Sayag, E., et al.: Simple compactifications and polar decomposition of homogeneous real spherical spaces. *Selecta Math. N.S.* **21**(3), 1071–1097 (2015)

[10] Knop, F.: The Luna–Vust theory of spherical embeddings. Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991

[11] Luna, D., Vust, Th., Plongements d'espaces homogènes. *Comment. Math. Helv.*, **58**, 186–245 (1983)

[12] Maclagan, D., Sturmfels, B.: Introduction to tropical geometry. Graduate Studies in Mathematics, AMS (2015)

[13] Nash, E., Tropicalizing spherical embeddings. arXiv:1609.07455

[14] Popov, V. L.: Contractions of actions of reductive algebraic groups. *Math. USSR-Sb.*, **58**(2), 311–335 (1987)

[15] Sakellaridis, Y.: Spherical varieties and integral representations of L-functions. *Algebra Number Theory*, **6**(4), 611–667 (2012)

[16] Sturmfels, B.: Gröbner bases and convex polytopes. University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996

[17] Sumihiro, H.: Equivariant completion. *J. Math. Kyoto Univ.*, **14**, 1–28 (1974)

[18] Tevelev, J.: Compactifications of subvarieties of tori. *Amer. J. Math.*, **129**(4), 1087–1104 (2007)

[19] Timashev, D.: Homogeneous spaces and equivariant embeddings. Encyclopaedia of Mathematical Sciences, 138. Invariant Theory and Algebraic Transformation Groups, 8. Springer, Heidelberg, 2011

[20] Vogiannou, T.: Spherical tropicalization. arXiv:1511.02203