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Global Phase Portraits of Quadratic Systems with a Complex Ellipse as Invariant Algebraic Curve

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Abstract In this paper, we study a new class of quadratic systems and classify all its phase portraits. More precisely, we characterize the class of all quadratic polynomial differential systems in the plane having a complex ellipse $x^2 + y^2 + 1 = 0$ as invariant algebraic curve. We provide all the different topological phase portraits that this class exhibits in the Poincaré disc.

Keywords Quadratic system, complex ellipse, invariant algebraic curves, phase portrait, Poincaré disc

MR(2010) Subject Classification 34C05

1 Introduction and Statement of the Main Results

A *planar polynomial differential system* is a differential system of the form

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{1.1}$$

where P and Q are real polynomials. The polynomial differential system (1.1) has *degree* n, if n is the maximum of the degrees of the polynomials P and Q. A *polynomial differential system of degree* 2 is called simply a *quadratic system*. The dot in system (1.1) denotes the derivative with respect to the independent variable t.

Many families of quadratic systems have been studied, and for them all the possible global topological phase portraits have been classified. One of the first of these classes studied was the classification of the quadratic centers and their first integrals which started with the works of Dulac [6], Kapteyn [9, 10], Bautin [3], Lunkevich and Sibirskii [14], Schlomiuk [22], Żołądek [26], Ye and Ye [25], Artés, Llibre and Vulpe [2], ... The class of the homogeneous quadratic

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systems, see Lyagina [15], Markus [16], Korol [11], Sibirskii and Vulpe [23], Newton [19], Date [5] and Vdovina [24], ... Another class is the one formed by the Hamiltonian quadratic systems, see Artés and Llibre [1], Kalin and Vulpe [8], and Artés, Llibre and Vulpe [2].

In this paper, we want to study a new class of quadratic systems and classify all its phase portraits. More precisely we analyze the class of all quadratic polynomial differential systems having a complex invariant ellipse, that after an affine transformation it can be written as

$$x^2 + y^2 + 1 = 0.$$

The family of quadratic systems having a real invariant ellipse has been studied by Qin [21] in 1958.

Our first result is to provide a normal form for all quadratic polynomial differential systems having a complex invariant ellipse

$$x^2 + y^2 + 1 = 0$$

as an invariant algebraic curve.

Proposition 1.1 *A planar polynomial differential system of degree* 2 *having a complex invariant ellipse* $x^2+y^2+1=0$ *as an invariant algebraic curve, after an affine change of coordinates, can be written as*

$$\begin{aligned} \dot{x} &= \alpha(x^2 + y^2 + 1) + y(ax + by + c), \\ \dot{y} &= -x(ax + by + c), \end{aligned} \tag{1.2}$$

where $\alpha, a, b, c \in \mathbb{R}$.

The proof of Proposition 1.1 is the same as the proof of the normal form of the quadratic system with a real invariant ellipse, see for more details the normal form given in system (12) of [13].

In the next theorem we present the topological classification of all the phase portraits in the Poincaré disc of planar polynomial differential systems of degree 2 having a complex invariant ellipse

$$x^2 + y^2 + 1 = 0$$

as an invariant algebraic curve. For a definition of the Poincaré compactification, the Poincaré disc and of topological equivalent phase portraits of a polynomial differential system in the Poincaré disc, see Section 2.

It is known that if a quadratic system has a limit cycle this surrounds a focus, see for more details [4]. Due to this fact we shall see that such quadratic systems have no limit cycles.

Theorem 1.2 *Given a planar polynomial differential system of degree* 2 *having a complex ellipse* $x^2+y^2+1=0$ *as an invariant algebraic curve its phase portrait is topological equivalent to one of the* L *phase portraits of Figure* 1.

Theorem 1.2 is proved in Section 3.

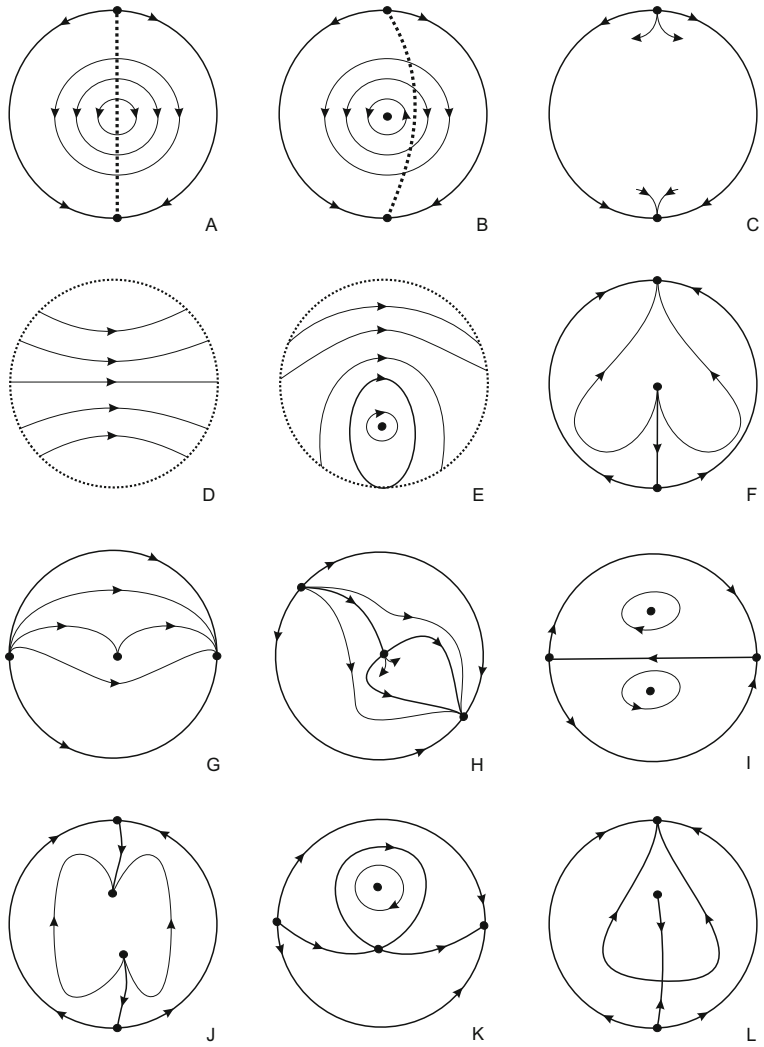

Figure 1 Phase portraits in the Poincaré disc of the quadratic systems having an invariant complex ellipse

2 Basic Results

2.1 Poincaré Compactification

The planar polynomial vector field of degree n associated to the polynomial differential system (1.1) of degree n is

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

The *Poincaré compactified vector field* $p(\mathcal{X})$ *associated to* $\mathcal{X}$ is an analytic vector field on $\mathbb{S}^2$ constructed as follows (see, for more details Chapter 5 of [7]).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the

tangent plane to $\mathbb{S}^2$ at point y. The tangent plane $T_{(0,0,1)}\mathbb{S}^2$ is identified with the plane $\mathbb{R}^2$ where we have the polynomial vector field $\mathcal{X}$. Consider the central projections $f_\pm : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. These two maps define two copies of $\mathcal{X}$, one $\mathcal{X}^+$ in the northern hemisphere and the other $\mathcal{X}^-$ in the southern hemisphere. Denote by $\mathcal{X}' = \mathcal{X}^+ \cup \mathcal{X}^-$ the vector field defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. The *infinity* of $\mathbb{R}^2$ is identified with the equator $\mathbb{S}^1$. The extension of $\mathcal{X}'$ to a vector field $P(\mathcal{X})$ on $\mathbb{S}^2$ (including $\mathbb{S}^1$) is the unique analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. Knowing the behaviour of $p(\mathcal{X})$ in a neighborhood of $\mathbb{S}^1$, we know the behaviour of $\mathcal{X}$ near the infinity. The Poincaré compactification is such that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

The *Poincaré disc*, denoted by $\mathbb{D}$, is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$.

Two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there is a homeomorphism on $\mathbb{S}^2$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$, preserving the infinity $\mathbb{S}^1$ and preserving or reversing simultaneously the sense of all orbits.

We consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ (which are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$ respectively in order to study the sphere $\mathbb{S}^2$ which is a differentiable manifold). If we denote by (u, v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so (u, v) represents different things according to the local charts under consideration), then we get for $p(\mathcal{X})$ the following expressions:

$$v^n \Delta(u,v)\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right), -vP\left(\frac{1}{v}, \frac{u}{v}\right)\right) \quad \text{in } U_1, \tag{2.1}$$

$$v^n \Delta(u,v)\left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right), -vQ\left(\frac{u}{v}, \frac{1}{v}\right)\right) \quad \text{in } U_2, \tag{2.2}$$

$$\Delta(u,v)(P(u,v), Q(u,v)) \quad \text{in } U_3,$$

where $\Delta(u, v) = (u^2+v^2+1)^{-\frac{1}{2}(n-1)}$. *In what follows we omit the factor* $\Delta(u, v)$ *by rescaling the vector field* $p(\mathcal{X})$. Thus we obtain a polynomial vector field in each local chart. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $v = 0$ always denotes the points of $\mathbb{S}^1$.

2.2 Separatrices and Canonical Regions

Let $p(\mathcal{X})$ be the Poincaré compactification in the Poincaré disc $\mathbb{D}$ of system (1.1), and let Φ be its analytic flow. Following Markus [17] and Neumann [18] we denote by (U, Φ) the flow of a differential system on an invariant set $U \subset \mathbb{D}$ under the flow Φ. Two flows (U, Φ) and (V, Ψ) are *topologically equivalent* if and only if there exists a homeomorphism $h : U \to V$ which sends orbits of the flow Φ into orbits of the flow Ψ either preserving or reversing the orientation of all the orbits.

A flow (U, Φ) is *parallel* if it is topologically equivalent to one of the following flows:

(i) The flow defined in $\mathbb{R}^2$ by the differential system $\dot{x} = 1$, $\dot{y} = 0$, called *strip flow*.

(ii) The flow defined in $\mathbb{R}^2 \setminus \{(0, 0)\}$ by the differential system in polar coordinates $\dot{r} = 0$,

$\dot{\theta} = 1$, called *annular flow.*

(iii) The flow defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ by the differential system in polar coordinates $\dot{r} = r$, $\dot{\theta} = 0$, called *spiral or radial flow.*

The separatrices of a vector field $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$ are:

(I) all the orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disc (i.e. at the infinity of $\mathbb{R}^2$),

(II) all the finite singular points of $p(\mathcal{X})$,

(III) all the limit cycles of $p(\mathcal{X})$, and

(IV) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

The vector fields $p(\mathcal{X})$, coming from polynomial vector fields (1.1) of $\mathbb{R}^2$ having finitely many singular points finite and infinite, have finitely many separatrices. For more details see for instance [12].

Let $\mathcal{S}$ be the union of all separatrices of the flow $(\mathbb{D}, \Phi)$ defined by $p(\mathcal{X})$ in the Poincaré disc $\mathbb{D}$. It is easy to check that $\mathcal{S}$ is an invariant closed set. If N is a connected component of $\mathbb{D} \setminus \mathcal{S}$, then N is also an invariant set under the flow Φ of $p(\mathcal{X})$, and the flow $(N, \Phi|_N)$ is called a *canonical region* of the flow $(\mathbb{D}, \Phi)$.

Proposition 2.1 *If the number of separatrices of the flow* $(\mathbb{D}, \Phi)$ *is finite, then every canonical region of the flow* $(\mathbb{D}, \Phi)$ *is parallel.*

For a proof of this proposition see [18] or [12].

The *separatrix configuration* $\mathcal{S}_c$ of a flow $(\mathbb{D}, \Phi)$ is the union of all the separatrices $\mathcal{S}$ of the flow together with an orbit belonging to each canonical region. The separatrix configuration $\mathcal{S}_c$ of the flow $(\mathbb{D}, \Phi)$ is topologically equivalent to the separatrix configuration $\mathcal{S}_c^*$ of the flow $(\mathbb{D}, \Phi^*)$ if there exists an orientation preserving homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ which transforms orbits of $\mathcal{S}_c$ into orbits of $\mathcal{S}_c^*$, and orbits of $\mathcal{S}$ into orbits of $\mathcal{S}^*$.

Theorem 2.2 (Markus–Neumann–Peixoto) *Let* $(\mathbb{D}, \Phi)$ *and* $(\mathbb{D}, \Phi^*)$ *be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields* (1.1)*. Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of this result see [17, 18, 20].

Theorem 2.2 implies that in order to classify the phase portraits in the Poincaré disc of a planar polynomial differential system having finitely many separatrices finite and infinite, it is enough to describe their separatrix configuration. This is what we have done in Figure 1.

3 Phase Portraits

It is clear that system (1.2) with $\alpha = 0$, after a rescaling of the time $\tau = (ax + by + c)t$ becomes

$$\dot{x} = y,$$
$$\dot{y} = -x,$$

whose phase portrait is formed by all the invariant circles centered at the origin of coordinates, providing the first two phase portraits of Figure 1.

In what follows we shall study the phase portraits of system (1.2) with $\alpha \neq 0$. Doing the rescaling of the time $\tau = \alpha t$, and renaming a/α, b/α, c/α again by a, b, c, respectively we have the quadratic system

$$\begin{aligned} \dot{x} =& x^2 + y^2 + 1 + y(ax + by + c), \\ \dot{y} =& -x(ax + by + c), \end{aligned} \tag{3.1}$$

with $a, b, c \in \mathbb{R}$.

Remark 3.1 We note that if we do the transformation $(x, y, t, c) \to (-x, -y, -t, -c)$ system (3.1) remains invariant. Hence, it is enough to consider only the cases in which $c \geq 0$.

Remark 3.2 System (3.1) with $a = 0$ is reversible because it does not change under the transformation $(x, y, t) \to (-x, y, -t)$. Hence we know that the phase portraits of system (3.1) with $a = 0$ are symmetric with respect to the y-axis.

The way for studying the phase portraits of systems (3.1) is the following. First we shall characterize the finite equilibria of those systems together with their local phase portraits. After we do the same for the infinite equilibria, and finally using this information on the equilibria and the existence of the invariant complex ellipse $x^2 + y^2 + 1 = 0$ we shall provide the classification of all the phase portraits of systems (3.1).

3.1 The Finite Singular Points

The finite singular points of system (3.1) are characterized in the next result.

Proposition 3.3 *System* (3.1) *has the following finite singular points*:

(a) *if* $b = -1$ *and* $c \neq 0$, *then*

$$U = \left(0, -\frac{1}{c}\right);$$

(b) *if* $b \neq -1$ *and* $c^2 = 4(b + 1)$, *then*

$$V = \left(0, -\frac{1}{\sqrt{b+1}}\right);$$

(c) *if* $b \neq -1$ *and* $c^2 > 4(b + 1)$, *then*

$$W_{\pm} = \left(0, -\frac{c \pm \sqrt{c^2 - 4(b+1)}}{2(b+1)}\right).$$

Proof The proof follows easily studying the real solutions of the system $x^2 + y^2 + 1 + y(ax + by + c) = 0$, $-x(ax + by + c) = 0$. □

We recall that $c \geq 0$. Now in the half plane (b, c) we define the regions

$$\begin{aligned} R_1 &= \{(b, c) : b < -1,\ c \geq 0\}, \\ R_2 &= \{(b, c) : b > -1,\ c > 2\sqrt{b+1}\}, \\ R_3 &= \{(b, c) : b > -1,\ 0 \leq c < 2\sqrt{b+1}\}, \end{aligned}$$

the curves

$$\begin{aligned} L_1 &= \{(b, c) : b = -1,\ c \neq 0\}, \\ L_2 &= \{(b, c) : b > -1,\ c = 2\sqrt{b+1}\}, \end{aligned}$$

and the point $P_1 = \{b = -1,\ c = 0\}$.

In view of Proposition 3.3, if the parameters (b, c) of system (3.1) are in R_3, then there are no singular points. Moreover, if the parameters (b, c) of system (3.1) correspond to the point P_1 then again there are no singular points.

From now on we only consider the remaining cases: L_1, L_2, R_1 and R_2.

For definitions of elliptic, parabolic and hyperbolic sectors, cusp, and hyperbolic, semi–hyperbolic and nilpotent singular points see [7].

We introduce some notation that will be used in the following result.

$$\begin{aligned} &\Delta = \sqrt{c^2 - 4(b+1)}, \quad D_{W_+} = -\frac{\Delta}{2(b+1)}(c(2+b) - b\Delta), \\ &T_{W_+} = \frac{a(-c-\Delta)}{2(1+b)}, \quad D_{W_-} = \frac{\Delta}{2(b+1)}(c(2+b) + b\Delta), \\ &T_{W_-} = \frac{a(-c+\Delta)}{2(1+b)}, \quad \mathrm{Dis}_\pm = T_{W_\pm}^2 - 4D_{W_\pm}. \end{aligned} \tag{3.2}$$

Proposition 3.4 *System* (3.1) *has the following finite singular points if its parameters* (b, c) *are in*

(L_1) *one singular point* U : *is a center if* $a = 0$; *an unstable hyperbolic node if* $a^2 \geq 4c^2(1+c^2)$ *with* $a < 0$; *a stable hyperbolic node if* $a^2 \geq 4c^2(1+c^2)$ *with* $a > 0$; *an unstable hyperbolic focus if* $a^2 < 4c^2(1+c^2)$ *with* $a < 0$, *and a stable hyperbolic focus if* $a^2 < 4c^2(1+c^2)$ *with* $a > 0$.

(L_2) *one singular point* V : *is a semi-hyperbolic saddle-node if* $a \neq 0$ *and a nilpotent cusp if* $a = 0$.

(R_1) *two singular points* W_+ *and* W_- : W_+ *is a center if* $a = 0$; *an unstable hyperbolic node if* $a > 0$ *and* $\mathrm{Dis}_+ \geq 0$; *a stable hyperbolic node if* $a < 0$ *and* $\mathrm{Dis}_+ \geq 0$; *an unstable hyperbolic focus if* $a > 0$ *and* $\mathrm{Dis}_+ < 0$ *and a stable hyperbolic focus if* $a < 0$ *and* $\mathrm{Dis}_+ < 0$; W_- *is a center if* $a = 0$; *an unstable hyperbolic node if* $a < 0$ *and* $\mathrm{Dis}_- \geq 0$; *a stable hyperbolic node if* $a > 0$ *and* $\mathrm{Dis}_- \geq 0$; *an unstable hyperbolic focus if* $a < 0$ *and* $\mathrm{Dis}_- < 0$, *and a stable hyperbolic focus if* $a > 0$ *and* $\mathrm{Dis}_- < 0$.

(R_2) *two singular points* W_+ *and* W_- : W_+ *is a hyperbolic saddle, and* W_- *is a center if* $a = 0$; *an unstable hyperbolic node if* $a < 0$ *and* $\mathrm{Dis}_- \geq 0$; *a stable hyperbolic node if* $a > 0$ *and* $\mathrm{Dis}_- \geq 0$; *an unstable hyperbolic focus if* $a < 0$ *and* $\mathrm{Dis}_- < 0$, *and a stable hyperbolic focus if* $a > 0$ *and* $\mathrm{Dis}_- < 0$.

Proof On the curve L_1 we have that $b = -1$ and $c \neq 0$. Clearly system (3.1) has the unique singular point U. The eigenvalues of the Jacobian matrix of system (3.1) at U are

$$\frac{-a + \sqrt{a^2 - 4c^2(1+c^2)}}{2c} \quad \text{and} \quad \frac{-a - \sqrt{a^2 - 4c^2(1+c^2)}}{2c}.$$

If $a = 0$, using the fact that system (3.1) is reversible with respect to the y-axis, and the eigenvalues of the Jacobian matrix are $\pm\,\mathrm{i}\sqrt{1+c^2}$ we get that U is a center. When $a \neq 0$, U is a hyperbolic node if $a^2 \geq 4c^2(1+c^2)$. Using that $c \geq 0$, it is unstable if $a < 0$ and stable if $a > 0$, see for more details Theorem 2.15 of [7] where are described the local phase portraits of the hyperbolic singular points. If $a^2 < 4c^2(1+c^2)$, since $a \neq 0$, then it is a hyperbolic focus, unstable if $a < 0$ and stable if $a > 0$.

On the curve L_2, we have $b > -1$ and $c = 2\sqrt{b+1}$. Clearly system (3.1) has the unique singular point V. The Jacobian matrix of system (3.1) at V is

$$J_V = \begin{pmatrix} -\dfrac{a}{\sqrt{1+b}} & 0 \\ -\dfrac{2+b}{\sqrt{1+b}} & 0 \end{pmatrix}.$$

Note that $b+2 \neq 0$ because $b > -1$. The eigenvalues of the Jacobian matrix of system (3.1) at V are 0 and $-a/\sqrt{1+b}$.

If $a \neq 0$ then it is a semi-hyperbolic point. We will use Theorem 2.19 of [7] for studying the local phase portraits of semi-hyperbolic singular points. Doing the change of variables $x = X$, $y = Y - \frac{1}{\sqrt{b+1}}$ the equilibrium V goes to the origin. Now the linear change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{2+b}{a} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

transforms the linear part of the system into its real Jordan form. In these new variables we get the system

$$\begin{aligned} u' &= -\frac{(1+b)(2+b)}{a}u^2 - \frac{2(1+b)(a^2+(2+b)^2)}{a^2}uv \\ &\quad - \frac{(a^2+(1+b)(2+b))(a^2+(2+b)^2)}{a^3}v^2, \\ v' &= \frac{a}{\sqrt{1+b}}v + (1+b)u^2 + \frac{a^2+2(1+b)(2+b)}{a}uv \\ &\quad + \frac{(1+b)(2+b)^2+a^2(3+b)}{a^2}v^2. \end{aligned}$$

Using Theorem 2.19 of [7] we get that V is a semi-hyperbolic saddle-node.

When $a = 0$ the V is a nilpotent singular point. We will use Theorem 3.5 of [7] for studying the local phase portraits of the nilpotent singular points. Again doing the change of variables $x = X$, $y = Y - \frac{1}{\sqrt{b+1}}$, and after the change $x = v$, $y = u$, and the rescaling of the independent variable $\tau = \frac{2+b}{\sqrt{1+b}}t$, we get the system

$$\begin{aligned} \dot{u} &= v + \frac{b\sqrt{1+b}}{2+b}uv, \\ \dot{v} &= -\frac{(1+b)^{3/2}}{2+b}u^2 - \frac{\sqrt{1+b}}{2+b}v^2. \end{aligned}$$

By Theorem 3.5 of [7] the singular point V is a cusp.

On the region R_1. System (3.1) has two singular points W_+ and W_-. We use the notation Δ in (3.2). Note that $\Delta > 0$. The Jacobian matrix of system (3.1) at W_+ is

$$J_{W_+} = \begin{pmatrix} -\frac{a(c+\Delta)}{2(1+b)} & -\Delta \\ -c + \frac{b(c+\Delta)}{2(1+b)} & 0 \end{pmatrix}. \tag{3.3}$$

The determinant of J_{W_+} is equal to D_{W_+} given in (3.2). Here $D_{W_+} > 0$. The trace of J_{W_+}, T_{W_+} is given in (3.2). Note that $-c - \Delta < 0$. So, $\operatorname{sign}(T_{W^+}) = \operatorname{sign}(a)$. By Remark 3.2, if $a = 0$ the system is reversible with respect to the y-axis and so W_+ is a center. If $\mathrm{Dis}_+ \leq 0$

(see (3.2)) then W_+ is a hyperbolic node (stable if $a < 0$ and unstable if $a > 0$), and if $\mathrm{Dis}_+ < 0$ then it is a hyperbolic focus (stable if $a < 0$ and unstable if $a > 0$).

The Jacobian matrix of system (3.1) at W_- is

$$J_{W_+} = \begin{pmatrix} -\frac{a(c-\Delta)}{2(1+b)} & \Delta \\ -c - \frac{b(c+\Delta)}{2(1+b)} & 0 \end{pmatrix}. \tag{3.4}$$

The determinant is equal to D_{W_-} given in (3.2). Here, again $D_{W_-} > 0$. The trace of J_{W_-} is T_{W_-} given in (3.2). Notice that since $\Delta > c$ then $\mathrm{sign}(T_{W_-}) = -\mathrm{sign}(a)$. If $a = 0$ the system is reversible with respect to the y-axis and so W_- is a center. If $\mathrm{Dis}_+ \leq 0$ (see (3.2)) then W_- is a hyperbolic node (stable if $a > 0$ and unstable if $a < 0$), and if $\mathrm{Dis}_+ < 0$ then it is a hyperbolic focus (stable if $a > 0$ and unstable if $a < 0$).

On the region R_2 the study can be done in a similar way to the study in the region R_1. Again, system (3.1) has the two singular points W_+ and W_-. The Jacobian matrix at W_+, J_{W_+} is given in (3.3) and its determinant, D_{W_+} is negative. So, W_+ is a hyperbolic saddle. The Jacobian matrix at W_-, J_{W_-} is given in (3.4) and its determinant, D_{W_-} is positive. The trace of J_{W_-} is T_{W_-}. So, if $a = 0$ then W_- is a center, and the other cases follow directly from the hyperbolic singular theorem, see Theorem 2.15 of [7]. □

3.2 The Infinite Singular Points

Proposition 3.5 *The following three statements hold.*

(a) *If $b \neq -1$, system* (3.1) *has a pair of infinite singular points which are hyperbolic saddles if $b < -1$, and hyperbolic stable nodes if $b > -1$;*

(b) *If $b = -1$ and $a = 0$ the infinity of system* (3.1) *is filled of singular points;*

(c) *If $b = -1$ and $a \neq 0$, system* (3.1) *has a pair of infinite singular points which are semi-hyperbolic. If $c > 0$ they are saddle-nodes, if $c = 0$ and $a < 0$ they are stable nodes, and if $c = 0$ and $a > 0$ they are unstable nodes.*

Proof First we study the infinite singular points of the local chart U_1. System (3.1) in the local chart U_1 using (2.1) is

$$\begin{aligned} \dot{u} &= -a - (1+b)u - cv - au^2 - (1+b)u^3 - cu^2v - uv^2, \\ \dot{v} &= -(1 + au + (1+b)u^2 + cuv + v^2)v. \end{aligned} \tag{3.5}$$

If $b \neq -1$, there is a unique infinite singular point of (3.5) which is $u = -a/(1+b)$, $v = 0$. The eigenvalues of the Jacobian matrix at this singular point are -1 and $-(a^2 + (1+b)^2)/(1+b)$, which implies that this singular point is a hyperbolic saddle if $b < -1$, and a hyperbolic stable node if $b > -1$. Since the origin of the local chart U_2 is not a singular point, the statement (a) follows.

If $b = -1$, from the study done for system (3.5) there are no infinite singular points in the local chart U_1. System (3.1) in the local chart U_2 using (2.2) is

$$\begin{aligned} \dot{u} &= au + cv + v^2 + au^3 + cu^2v, \\ \dot{v} &= uv(-1 + au + cv). \end{aligned} \tag{3.6}$$

If $a = 0$, then the infinity is filled by singular points.

If $a \neq 0$, the origin is a singular point of (3.6). It is easy to get that the eigenvalues of the Jacobian matrix at the origin are 0 and a, which implies that the origin is a semi-hyperbolic singular point. Using Theorem 2.19 of [7] we get that if $c \neq 0$ then the origin is a semi-hyperbolic saddle-node. If $c = 0$ and $a < 0$ the origin is a semi-hyperbolic stable node, and if $c = 0$ and $a > 0$ the origin is a semi-hyperbolic unstable node. □

Proof of Theorem 1.2 Using Propositions 3.4 and 3.5, and the existence of some first integrals, we shall obtain the global phase portraits of system (3.1) in Poincaré disc described in Figure 1.

For P_1 if $a \neq 0$ the phase portrait is given in C of Figure 1, and if $a = 0$ then it is given in the D of Figure 1. We note that the phase portrait C follows from the local study of the finite and infinite singular points done in Propositions 3.4 and 3.5 together with the Bendixson–Poincaré Theorem, see for instance [7]. The phase portrait D follows from the fact that on P_1 the system has the first integral $H = y^2/(1 + x^2)$ and all the orbits satisfy $H =$ constant.

For L_1 if $a = 0$ the phase portrait is given in E of Figure 1, and if $a \neq 0$ then it is given in F of Figure 1. The phase portrait E follows from the fact that on L_1 the system has the first integral $H = (c - y)^2/(1 - c^2 + x^2 + 2cy)$. The phase portrait F follows from the local study of the finite and infinite singular points done in Propositions 3.4 and 3.5 together with the Bendixson–Poincaré Theorem.

For L_2 if $a = 0$ the phase portrait is given in G of Figure 1, and if $a \neq 0$ then we get the phase portrait H of Figure 1. Both phase portraits follow from the local study of the finite and infinite singular points done in Propositions 3.4 and 3.5 together with the Bendixson–Poincaré Theorem.

For R_1 if $a = 0$ the phase portrait is given in I of Figure 1, and if $a \neq 0$ then we get the phase portrait H of Figure 1. The phase portrait I follows from the fact that on R_1 the system has the first integral $H = (c + by)^2(1 + x^2 + y^2)^b$. The phase portrait H follows from the local study of the finite and infinite singular points done in Propositions 3.4 and 3.5 together with the Bendixson–Poincaré Theorem.

For R_2 if $a = 0$ the phase portrait is given in K of Figure 1, and if $a \neq 0$ then it is given in L of Figure 1. The phase portrait K follows from the fact that on R_2 the system has the first integral $H = (c + by)^2(1 + x^2 + y^2)^b$. The phase portrait L follows from the local study of the finite and infinite singular points done in Propositions 3.4 and 3.5 together with the Bendixson–Poincaré Theorem.

For R_3 the phase portrait is given in C of Figure 1. This completes the proof of Theorem 1.2.

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